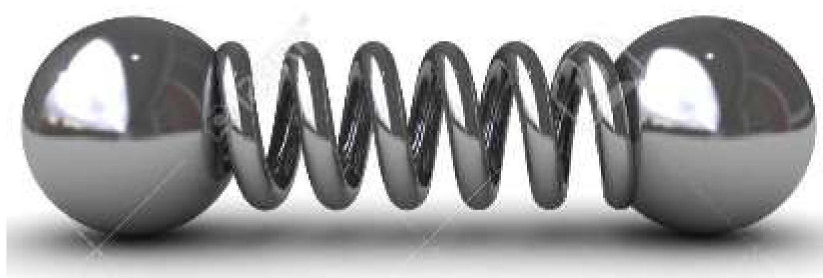
Chapter 3

Elastic Properties of Material

Stress and Strain



States of Matter

- Solid
- Liquid
- Gas
- Plasmas

Solids: Stress and Strain

Stress = Measure of force felt by material

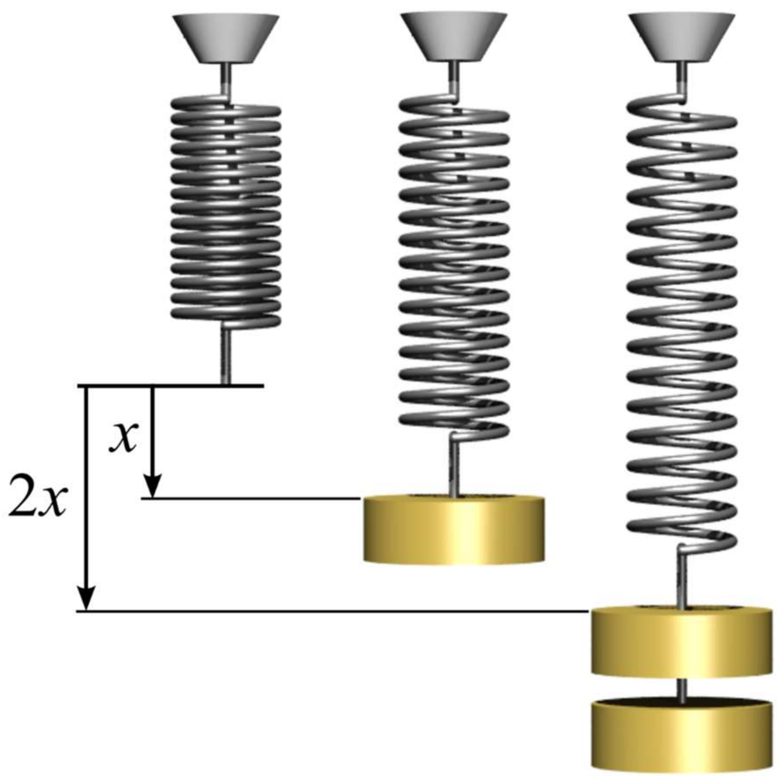
$$Stress\ (\sigma) = \frac{Force}{Area} = \frac{F}{A}$$

- **SI units are Pascals, 1 Pa = 1 N/m^2**
  (same as pressure)

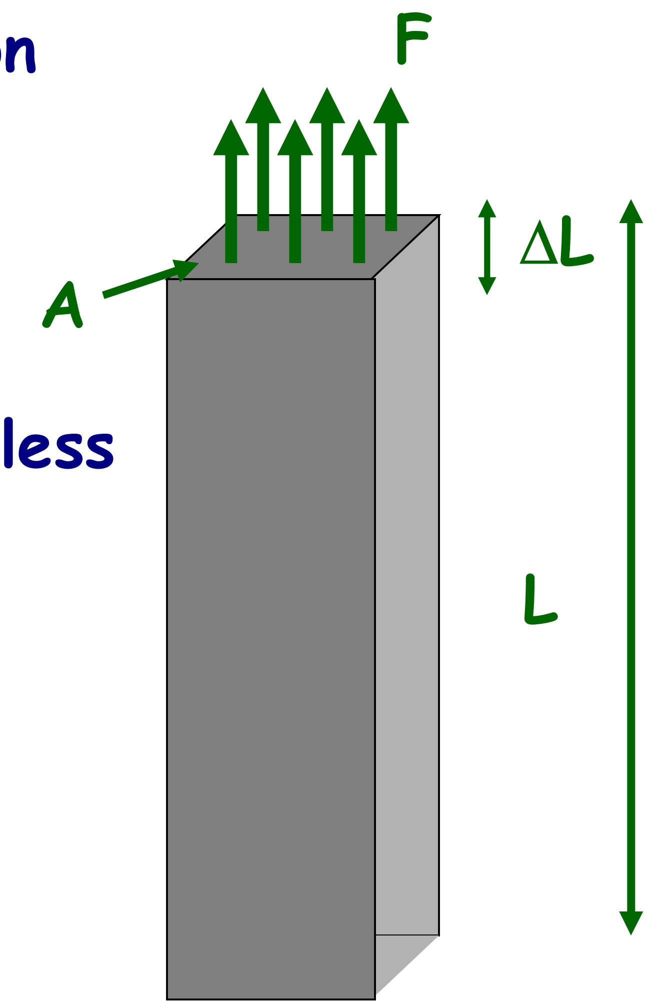
Solids: Stress and Strain

Strain = Measure of deformation

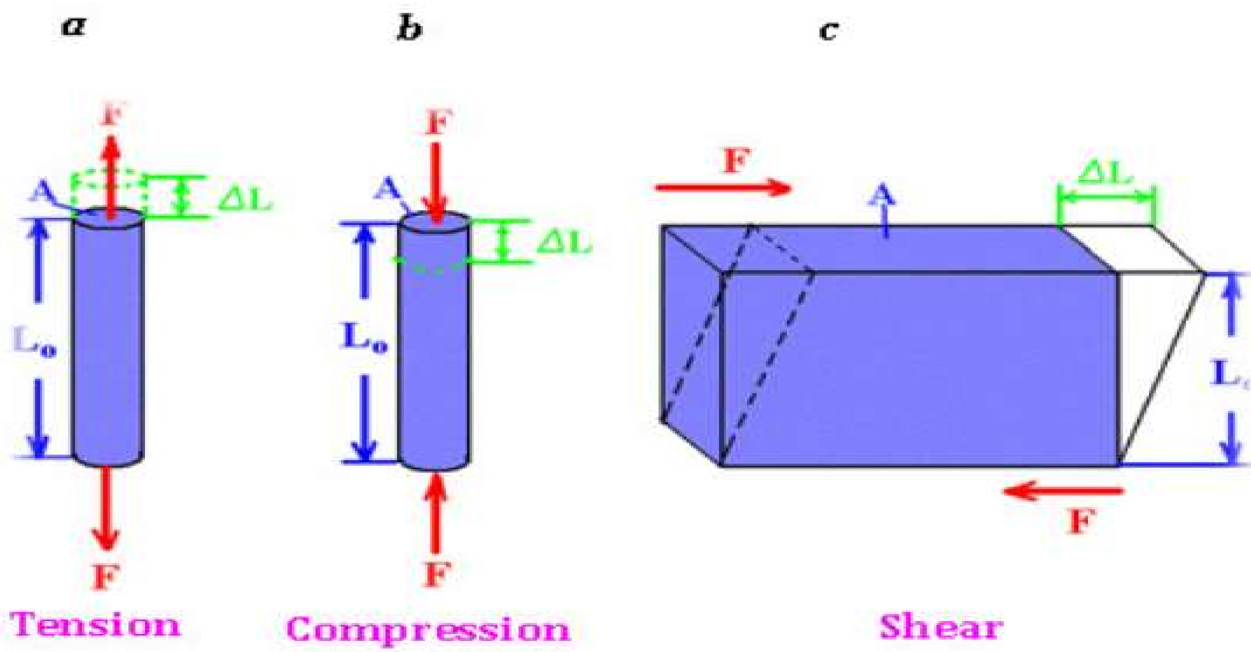
$$Strain = \frac{\Delta L}{L}$$



- **dimensionless**



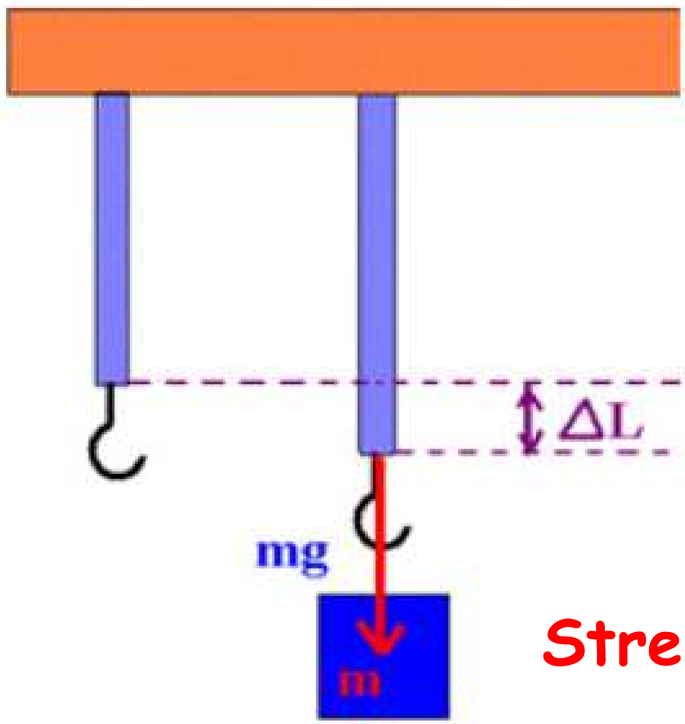
There are three kinds of stress commonly defined as:

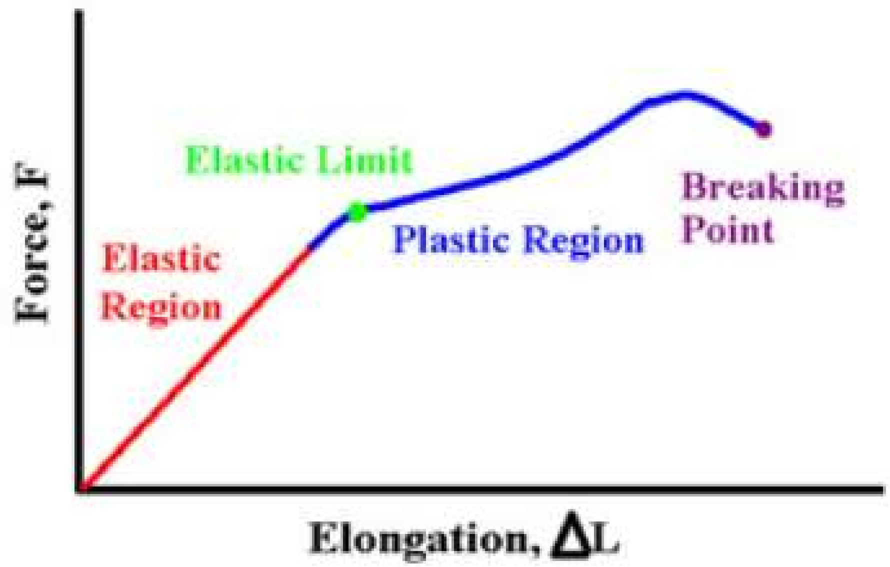


extension as a fractional change in length. We call this the **tensile strain** which we define as **the extension per unit length**.

$$\varepsilon = \frac{\Delta L}{l} \tag{3.2}$$

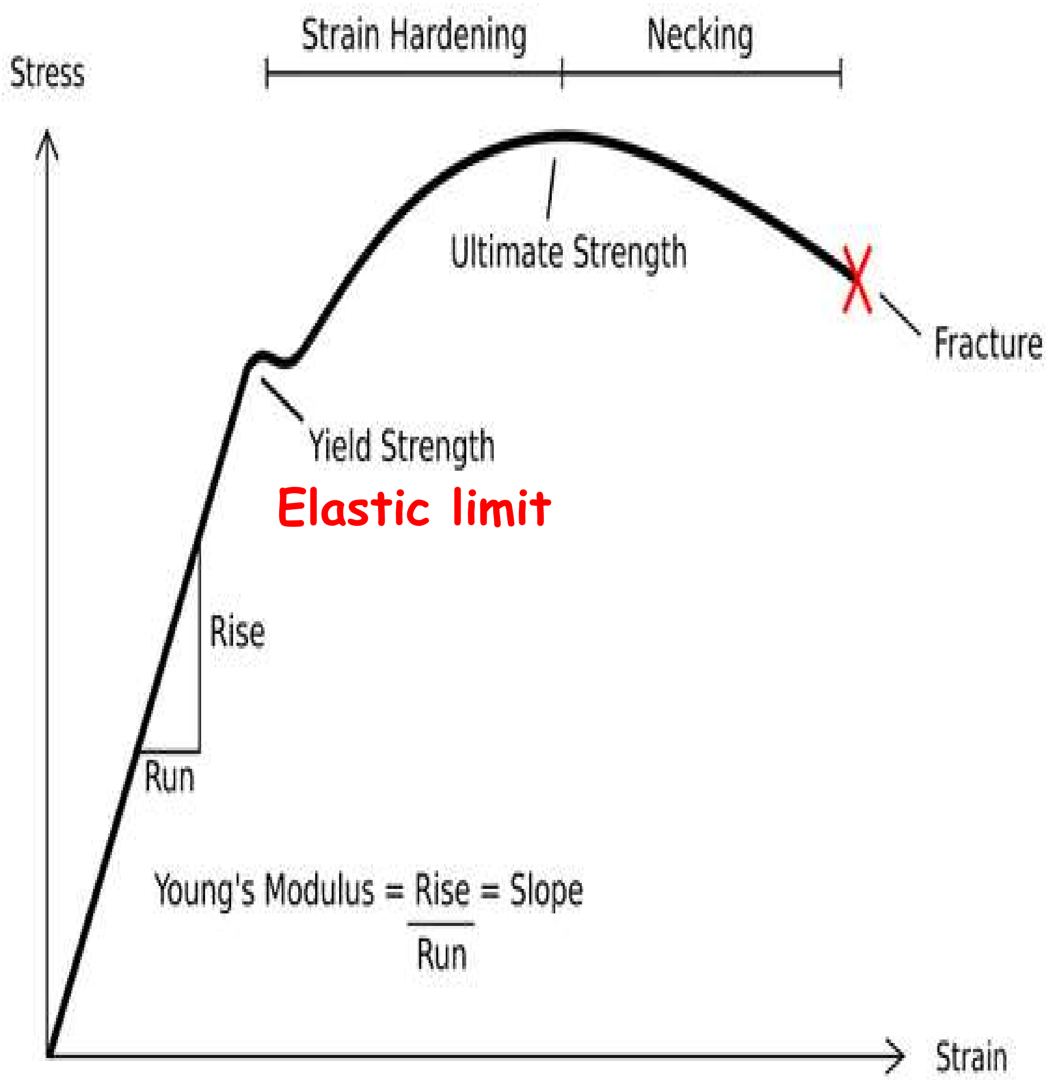
Materials Beaviour under Stress





Stress (σ) = F/A

Elastic Region and Elastic Limit
Plastic Region Permanent Deformation
Ultimate strength
Fracture Point



Stress (σ) = F/A

Strain (ε) = ΔL / L

$$Y = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta L/L} = \frac{F\,L}{A\,\Delta L}$$

Linear Region (Elastic Region)

3.2 Young modulus

The form of Hooke's law represented in Eq. 3.3 is not the most useful form because the value of ***k*** is different not just for each material but also for different cross-sectional areas. So we need to introduce a new quantity known as Young modulus or elastic modulus, which depends only on the material type. Young modulus relates the elastic deformation of a solid to the associated stresses. It is defined ***as the ratio of stress over strain in the region in which Hooke's Law is obeyed for the material.*** This can be experimentally determined from the slope of a stress-strain curve created during tensile tests conducted on a sample of the material.

$$Y = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta L/L} = \frac{F\,L}{A\,\Delta L}$$

Units N/m², measure of the hardness (Stiffness) of the material

$$Y = \frac{\sigma}{\varepsilon}$$

$$Y = \frac{F/A}{\Delta l/l}$$

From which one can find

$$F = \frac{YA}{l}\Delta l$$

$F = k\,\Delta L$

K is the constant is called spring constant

K= YA/L measured in N/m

A 15 cm long animal tendon was found to stretch 3.7 mm by a force of 13.4 N. The tendon was approximately round with an average diameter of 8.5 mm. Calculate the elastic modulus of this tendon

$$l_0 = 15 \times 10^{-2}\, m$$

$$\Delta l = 3.7 \times 10^{-3}\, m \quad , \quad F = 13.4\, N$$

$$A = \pi r^2 = 3.14 \times \left(\frac{8.5}{2} \times 10^{-3}\right)^2$$

$$Y = \frac{F l_0}{A \Delta l} = \frac{\cancel{3.14} \times 15 \times 10^{-2}}{\cancel{3.14} \times (4.25 \times 10^{-3})^2 \times 3.7 \times 10^{-3}}$$

$$\boxed{Y = 2.2 \times 10^{8}\, N/m^2}$$

Values of Young´s Modulus for some Materials

Material	*Young's modulus (N m^{-2})*
Steel	2.0×10^{11}
Brass	1.3×10^{11}
Cast iron	1.0×10^{10}
Aluminium	7.0×10^{10}
Marble	5.0×10^{10}
Concrete	2.0×10^{10}
Brick	1.4×10^{10}
Bone (human femur)	1.4×10^{10}
Timber (pine)	
parallel to the grain	1.0×10^{10}
across the grain	1.0×10^{9}
Nylon	5.0×10^{9}
Glass (crown)	7.1×10^{10}
Granite	4.5×10^{10}
Rubber	4.0×10^{6}

Example

(a) Find the minimum diameter of a steel wire 18 m long that elongates no more than 9 mm when a load of 380 kg is hung on its lower end. (b) If the elastic limit for this steel is $3 \times 10^{8}\ \mathrm{N/m^2}$, does permanent deformation occur with this load?

Steel wire , $l_0 = 18m$, $\Delta l = 9 \times 10^{-3} m$

load $F = 3800 N$

Find the minimum diameter

rom the table Young's Modulus for steel $= 2 \times 10^{11}$ N/m

stress $\sigma = Y \varepsilon = Y \frac{\Delta l}{l_0}$

$$\frac{F}{A} = Y \frac{\Delta l}{l_0} \Rightarrow A = \frac{F l_0}{Y \Delta l}$$

$$\frac{F}{A} = Y \frac{\Delta l}{l_0} \Rightarrow A = \frac{F l_0}{Y \Delta l}$$

where $A = \pi r^2 = \frac{F l_0}{Y \Delta l}$

$$\therefore r^2 = \frac{F l_0}{\pi Y \Delta l} = \frac{3800 \cdot 18}{3.14 \times 2 \times 10^{11} \times 9 \times 10^{-3}}$$

$$r^2 = 12.1 \times 10^{-6} \Rightarrow r = 3.47 \times 10^{-3} \, m$$

eter $\Rightarrow$ $d = 6.95 \times 10^{-3} \, m \approx 7 \, mm$

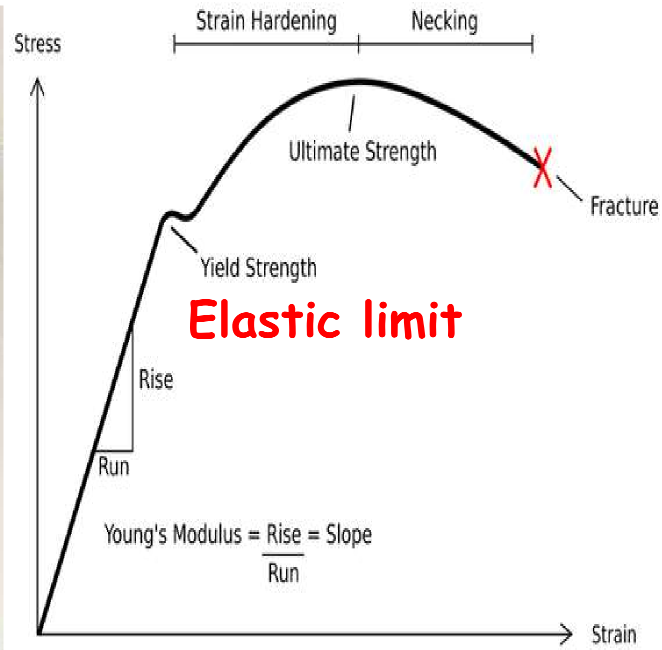
If the elastic limit for this steel is $3 \times 10^{8}\ \mathrm{N/m^2}$, does permanent deformation occur with this load?

σ = Stress applied $= \dfrac{F}{A} = \dfrac{3800}{3.14 \times (3.47 \times 10^{-3})^2}$

$\sigma = 10^{8}\ N/m^2$

elastic limit $\sigma_{el} = 3 \times 10^{8}\ N/m^2$

$\therefore$ $\sigma < \sigma_{el}$, so the deformation is temporary $\boxed{\text{[illegible]}}$



7. A pipe has an inner radius of $0.02\ m$ and an outer radius of $0.023\ m$. If subjected to a tension stress of $5 \times 10^7 \mathrm{Nm}^{-2}$, how large is the applied force?

$\sigma = 5 \times 10^7 \ N/m^2$

$F = \sigma A$

$A = \pi (r_{out}^2 - r_{in}^2)$

$r_{in} = 0.02$

$r_{out} = 0.023$

8. A man leg can be thought of as a shaft of bone $1.2\,m$ long. If the strain is 1.3×10^{-4} when the leg supports his weight, by how much is his leg shortened?

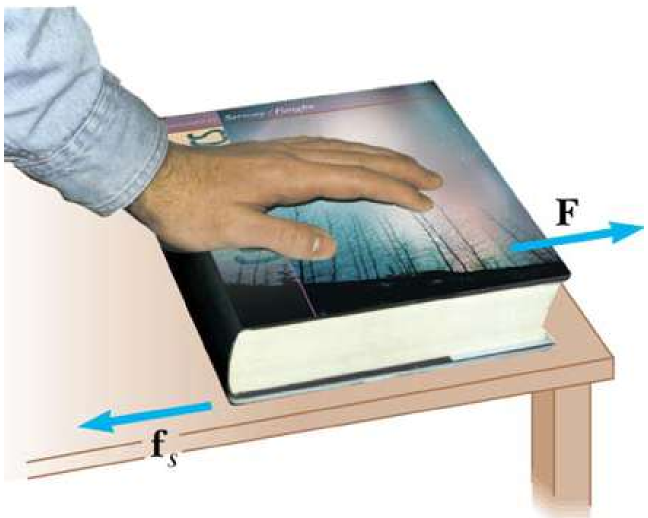
$l_0 = 1.2\,m$, strain $\varepsilon = \frac{\Delta l}{l_0} = 1.3 \times 10^{-4}$

Find Δl

$$\frac{\Delta l}{l_0} = \varepsilon_0 \Rightarrow \Delta l = l_0 \varepsilon$$

$$\Delta l = 1.2 \times 1.3 \times 10^{-4} = 1.56 \times 10^{-4}\,m = 156\,\mu m$$

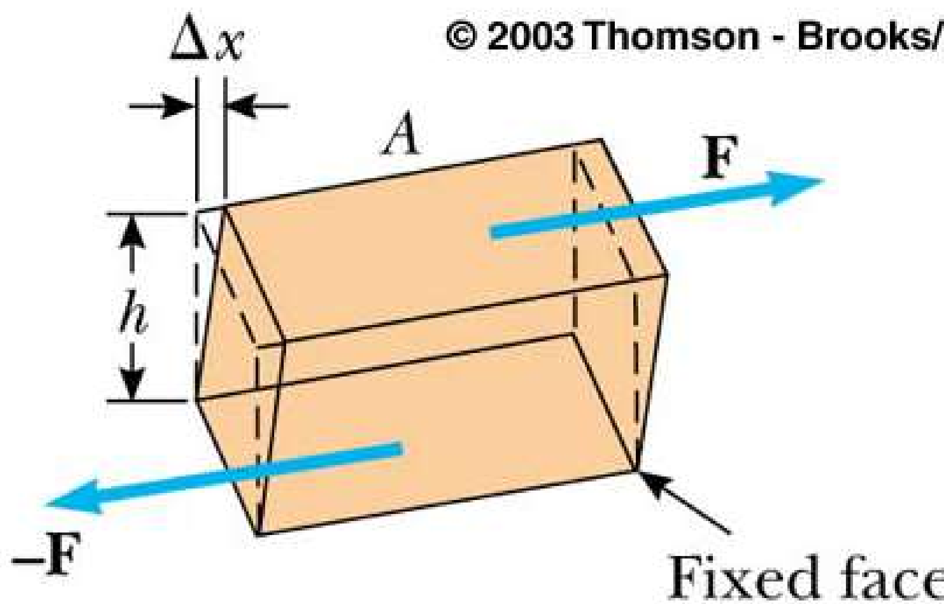
Shear Modulus



$$S = \frac{\left(F/A\right)}{\left(\Delta x/h\right)}$$

Sheer Stress

Sheer Strain



Stress in 3D

Bulk Modulus

$$B = -\frac{\Delta F / A}{\Delta V / V} = -\frac{\Delta P}{\left(\Delta V / V\right)}$$

Change in Pressure

Volume Strain

$$B = Y/3$$

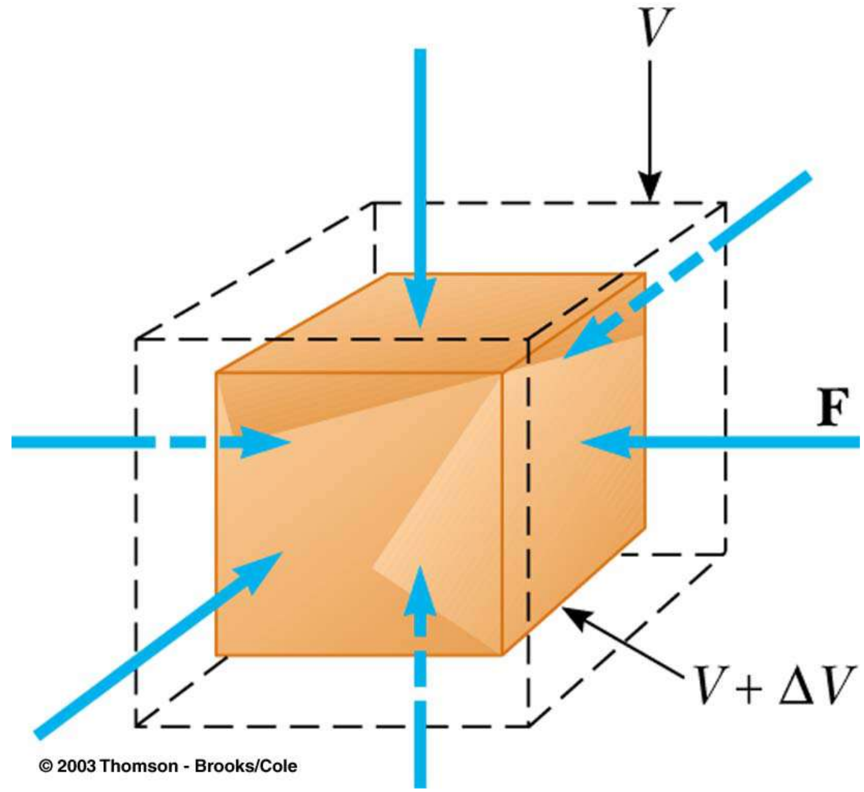
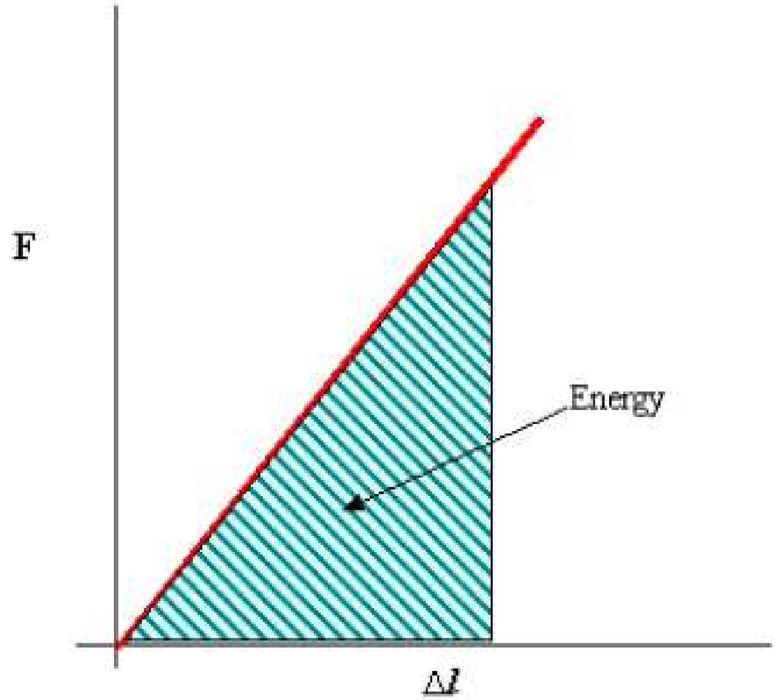


Table3.1: typical values for elastic Modulus

substance	young modulus(N/m^2)	Shear modulus(N/m^2)	bulk modulus(N/m^2)
Tungsten			
Steel			
Copper	11×10^{10}	4.2×10^{10}	14×10^{10}
Brass	9.1×10^{10}	3.5×10^{10}	6.1×10^{10}
Aluminum	7×10^{10}	2.5×10^{10}	7×10^{10}
Glass	$6.5 - 7.8 \times 10^{10}$	$2.6 - 3.2 \times 10^{10}$	$5 - 5.5 \times 10^{10}$
Quartz	5.6×10^{10}	2.6×10^{10}	2.7×10^{10}
Water	–	–	0.21×10^{10}

Elastic Strain Energy

When we stretch a wire, a work done on the wire. We are stretching the bonds between the atoms. If we release the wire, we can recover the energy stored in the wire due to stretch, which is called the **elastic strain energy**.



$$\mathcal{E} = \frac{1}{2} F \Delta l$$

Substituting for F from eq. 3.5 we get

$$F = \frac{YA\Delta l}{l}$$

$$\mathcal{E} = \frac{YA}{2l} \Delta l^2$$

Bone Fracture, Energy consideration

$$F_B = \sigma_B A = \frac{YA}{l}\Delta l$$

The compression Δl at the breaking point is, therefore,

$$\Delta l = \frac{\sigma_B l}{Y}$$

From Eq. 3.8, the energy stored in the compressed bone at the point of fracture is

$$\mathcal{E} = \frac{YA}{2l}\left(\frac{\sigma_B l}{Y}\right)^2$$

$$\boxed{\mathcal{E} = \frac{Al{\sigma_B}^2}{2Y}}$$

Example 3.6

Discuss the possibility of fracture of two leg bones that each have a length of about 90 cm and an average area of about 6 cm2 when a 70 kg person jump from a height of 60 cm.

The breaking stress of the bone σ_B is $10^9\ dyn/cm^2$, and Young's modulus for the bone is $14\ 10^{10}\ dyn/cm^2$. $(Hint: dyn = 10^{-5}N = 10\mu N, erg = 10^{-7}J,$ $dyn/cm^2 \equiv 10^{-1}\ N/m^2)$

The total energy absorbed by the bones of one leg at the point of compressive fracture is, from Eq. 3.11,

$$\mathcal{E} = \frac{1}{2}\frac{Al{\sigma_B}^2}{Y}$$

$$= \frac{1}{2}\frac{(6 \times 10^{-4}m^2)(0.9m)(10^8N/m^2)^2}{14 \times 10^9N/m^2} = 192.86J$$

Now we have to calculate the energy gained by jumping

$$\mathcal{E} = mgh = 70\ (9.8)(0.60) = 411.6\ J$$

Now, this energy is evenly distributed on both legs, thus each leg absorbs energy equals to $411.6J/2 = 205.8J$. This number exceeds the allowed energy to be stored at the bone break which will endanger the leg bones and it would cause bone fracture. If this person jumped on one leg this might lead to multiple fracture because of the huge difference between the allowed energy (less than $192.86J$) and the energy stored due to jumping ($411.6J$).

Examples

16. Assume that a 50 kg runner trips and falls on his extended hand. If the bones of one arm absorb all the kinetic energy (neglecting the energy of the fall), what is the minimum speed of the runner that will cause a fracture of the arm bone? Assume that the length of arm is 1 m and that the area of the bone is 4 cm^2.

$m = 50\,kg$, $A = 4\,cm^2 = 4\times10^{-4}\,m^2$, $l = 1\,m$

$E_k \equiv$ Kinetic energy $= \frac{1}{2} m \upsilon^2 = \frac{1}{2}(50)\upsilon^2 = 25\upsilon^2$

$$\mathcal{E}_{fracture} = \frac{A l \sigma_B^2}{2Y} = \frac{4\times10^{-4}\times1\times(10^8)^2}{2\times14\times10^9} = 142.86\,J = 25\upsilon^2$$

$$\upsilon^2 = \frac{142.86}{25} \Rightarrow \upsilon = 5.7\,m/s$$

A child slides across a floor in a pair of rubber-soled shoes. The frictional force acting on each foot is 20 N. The footprint area of each shoe's sole is 14 cm^2, and the thickness of each sole is 5 mm. Find the horizontal distance by which the upper and lower surfaces of each sole are offset. The shear modulus of the rubber is $3 \times 10^6\ \mathrm{N/m^2}$.

$$S = \frac{F/A}{\Delta x/h} \rightarrow\rightarrow \Delta x = \frac{Fh}{AS} = \frac{20\,(5\times10^{-3})}{(14\times10^{-4})(3\times10^{6})}$$

$$= \frac{10^{-3}}{42} = 2.4 \times 10^{-5} m$$