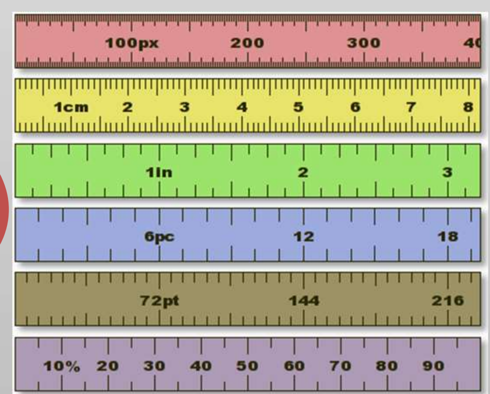
Lecture 1
Quantities and Dimensions

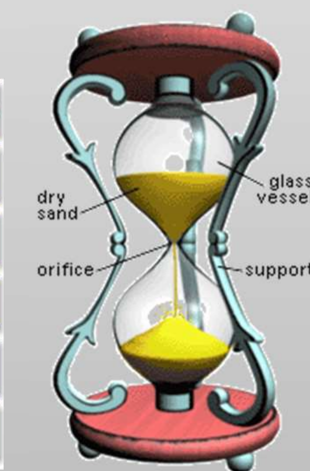
Type Quantities

- **Choose three basic quantities** **(Fundamentals):**
  - **LENGTH**
  - **MASS**
  - **TIME**
- **Define other units in terms of these** **(Derivatives)**
- ***Speed , density, force, work, etc..***
- ***speed=Length/time,***
- ***density= mass/ (length)^3***
- ***force= mass x acceleration = mass x length/(time)^2***
- ***Work = force x distance = mass x (length)^2/(time)^2***

Unit for 3 Basic Quantities

- **Many possible choices for units of Length, Mass, Time (Fundamental Units)**
- **In 1960, standards bodies control and define Système Internationale (SI) unit as, or mks**
  - **LENGTH: Meter (m)**
  - **MASS: Kilogram (kg)**
  - **TIME: Second (s)**





Fundamental and Derivative Quantities and SI Units

F

D

Length	meter	m
Mass	kilogram	kg
Time	second	s
speed	Meter/secon d	m/s
Force	Newton (N)	Kg m/s^2
density	Kg/m^3	Kg/m^3
work	Joule	Kgm^2/s^2

Approximate values of some measured lengths

Distance from earth to most remote known star	1×10^{26} meters
Distance from Earth to Andromeda galaxy	2×10^{22} m
One light year	9×10^{15} m
Orbit radius of Earth about sun	2×10^{11} m
Mean distance from Earth to moon	4×10^{8} m
Length of a football field	9×10^{1} m
Length of a housefly	5×10^{-3} m
Size of smallest dust particles	1×10^{-4} m
Size of most living cells	1×10^{-5} m
Diameter of hydrogen atom	1×10^{-10} m
Diameter of atomic nucleus	1×10^{-14} m
Diameter of a proton	1×10^{-15} m

Approximate values of some masses

Observable Universe	1×10^{52} kilograms
Earth	6×10^{24} kilograms
Shark	1×10^{2} kilograms
Human	7×10^{1} kilograms
Mosquito	1×10^{-5} kilograms
Bacterium	1×10^{-15} kilograms
Hydrogen Atom	2×10^{-27} kilograms
Electron	9×10^{-31} kilograms

Prefixes for SI Units

- 3,000 m = 3 x1,000 m
  = 3 x 10^3 m = 3 km
- 1,000,000,000 = 10^9 = 1G
- 1,000,000 = 10^6 = 1M
- 1,000 = 10^3 = 1k

- 141 kg = ? g
- 1 GB = ? Byte = ? MB

10^x	Prefix	Symbol
x=18	exa	E
15	peta	P
12	tera	T
9	giga	G
6	mega	M
3	kilo	k
2	hecto	h
1	deca	da

Prefixes for SI Units

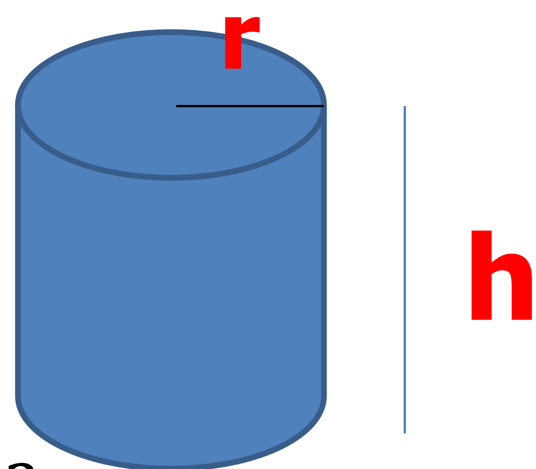
10^x	Prefix	Symbol
x=-1	deci	d
-2	centi	c
-3	milli	m
-6	micro	μ
-9	nano	n
-12	pico	p
-15	femto	f
-18	atto	a

Other Unit System

- U.S. customary system: foot, slug, second
- Cgs system: cm, gram, second
- We will use SI units in this course, but it is useful to know conversions between systems.
  - 1 mile = 1609 m = 1.609 km , 1 ft = 0.3048 m = 30.48 cm
  - 1 m = 39.37 in. = 3.281 ft , 1 in. = 0.0254 m = 2.54 cm
  - 1 lb = 0.465 kg 1 oz = 28.35 g 1 slug = 14.59 kg
  - 1 day = 24 hours = 24 * 60 minutes = 24 * 60 * 60 sec

Unit Conversion

A cylinder of 12 cm in height and its diamer is 40 mm, calculate The density of its material in SI unit, if its mass is 870 g?



$$\text{The density} = \frac{mass}{volume} = \frac{m}{\pi r^2 h}$$

$$= \frac{870\times10^{-3}\ kg}{3.14\times(20\times10^{-3})^2 m^2\ (12\times10^{-2})m}$$

$$\cong$$

$$6 \times 10^3\ kg/m^3$$

Dimensions, Units and Equations

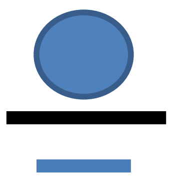
- Quantities have dimensions:
  - **Length – L, Mass – M, and Time - T**
- Quantities have units: Length – m, Mass – kg, Time – s
- To refer to the dimension of a quantity, use square brackets, e.g. $[F]$ means dimensions of force.

Quantity	Area	Volume	Speed	Acceleration
Dimension	$[A]$ = L^2	$[V]$ = L^3	$[v]$ = L/T	$[a]$ = L/T^2
SI Units	m^2	m^3	m/s	m/s^2

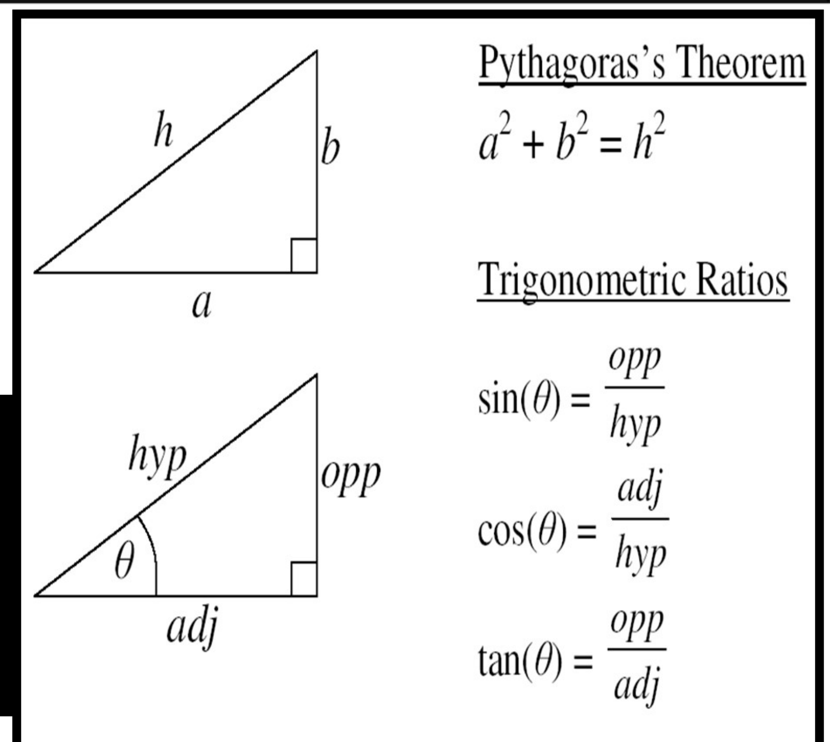
Dimensionless quantity

is a quantity that has no unit and it is a ratio between two similar quantities, to which no physical dimension is assigned. It is also known as a bare number or pure number or a quantity of dimension one [1]

π



$$\text{Specific Gravity} = \frac{\text{density of the object}}{\text{density of water}} = \frac{\rho_{object}}{\rho_{H_2O}}$$



Quantity	Description	Dimension	Units
Velocity	$Displacement / Time$	$[v] = [LT^{-1}]$	m/sec
Acceleration	$Displacement / Time^2$	$[a] = [LT^{-2}]$	m/sec2
Momentum	$Mass \times Velocity$	$[p] = [MLT^{-1}]$	Kg m/sec
Force	$Momentum / Time$	$[F] = [MLT^{-2}]$	Newton
Pressure	$Force / Area$	$[p] = [ML^{-1}T^{-2}]$	Poise

Why Dimensional Analysis is Important

- ❑ Help to understand Physics , even if you are not Physicst
- ❑ Check your equation or Formula (Correct or not?!!!)
- ❑ Derive relation between physical quantities in
- ❑ physical phenomena
  - ✓ Equations must be dimensionally consistent?????
    - ❖ Means that each term in the equation have the same dimension

$$A + B = C + D$$

A dimension equation is said to be consistent, only and only if dimensions of equations are same on both sides. Also, remember that if an equation is dimensionally correct it doesn't mean it is a

Check the consistency of the equation

$x = x_0 + v_0t + (1/2)\ at$

where x and x_0 are distances, t is time, v is velocity and a is an acceleration of the body.

Now to check if the above equation is dimensionally correct, we have to prove that dimensions of physical quantities are same on both sides. Also, we have to keep in mind that quantities can only be added or subtracted if their dimensions are same.

x = distance = [L]

x_0= distance = [L]

v_0t = velocity × time = $[LT^{-1}] \times [T] = [L]$

at = acceleration × time = $[LT^{-2}] \times [T] = [LT^{-1}]$

Since dimensions of at is not equals to the dimension of the other terms in the, so the equation is said to be inconsistent and dimensionally incorrect.

Example

Check whether the given equation is dimensionally correct.
$W = 1/2\ mv^2 - mgh$
where W stands for work done, m means mass, g stands for gravity, v for velocity and h for height.

To check the above equation as dimensionally correct, we first write dimensions of all the physical quantities mentioned in the equation.

W = Force × Displacement $= [MLT^{-2}] \times [L] = [ML^2T^{-2}]$

$1/2\ mv^2$ = Kinetic Energy $= [M] \times [L^2T^{-2}] = [ML^2T^{-2}]$

mgh = Potential Energy $= [M] \times [LT^{-2}] \times [L] = [ML^2T^{-2}]$

Since all the dimensions on left and right sides are equal it is a dimensionally correct equation.

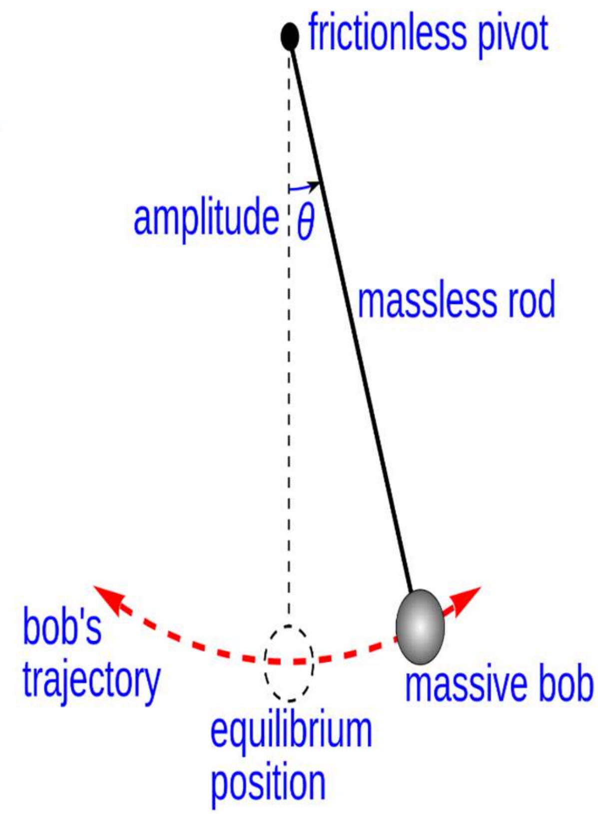
Use of DA

Derive relations between physical quantities involved in physical phenomena

The method of dimension analysis can also help in finding out relations between physical quantities. If we know how physical quantities depend on each other, we can find the relation between them easily by equating dimension on both sides.

Example

Suppose a bob is hanging from a ceiling (simple Pendulum) and time period of oscillations depends on 'length "l" of the thread, mass "m" of the bob and gravity "g". In order to find a relation between time and other physical quantities, we proceed as follows:



Let's time depends on powers x, y and z of length l, mass m and gravity g of the bob. Then the equation becomes

$T = k\, l^x m^y g^z$, where k is a proportionality constant

Writing dimensions on both sides, we get

Arranging powers accordingly we get

$[[M^0L^0T] = k\, [L]^x[M]^y[LT^{-2}]^z$

$M^0L^0T] = k\, [M^yL^{x+z}T^{-2z}]$

Equating powers on both sides, we get three equations

$y = 0$, $x + z = 0$, $-2z = 1$, Solving the linear equations we get

$x = 1/2$, $y = 0$ and $z = -1/2$

Substituting the values of x, y and z in the equation we have derived the following relation:

$T = k\, L^{1/2}M^0g^{-1/2} \quad = k\sqrt{\frac{l}{g}}$

Example 1.6

Suppose we are told that the acceleration **a** of a particle moving with uniform speed **v** in a circle of radius r is proportional to some power of r, say r^n, and some power of **v**, say $\mathbf{v}^m$ How can we determine the values of n and m?

Example 1.7

If the frequency (f) of a simple harmonic motion of a pendulum can be written as,

$$f = m^{\alpha} g^{\beta} l^{\gamma}$$

where m is the mass of the oscillating pendulum, g is the acceleration due gravity, and l is the length of the pendulum. Find α, β, and γ.

Vector and Scalar Quantities

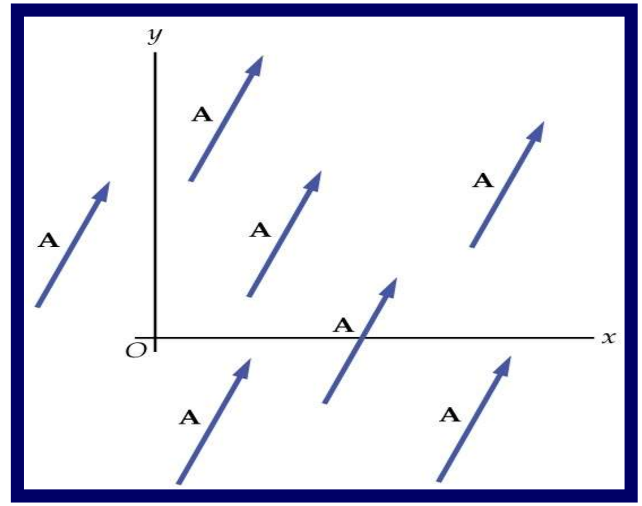
- Vectors: magnitude + direction
  - Displacement
  - Velocity (magnitude and direction!)
  - Acceleration
  - Force
  - Momentum
- Scalars: magnitude
  - Distance
  - Speed (magnitude of velocity)
  - Temperature
  - Mass
  - Energy
  - Time

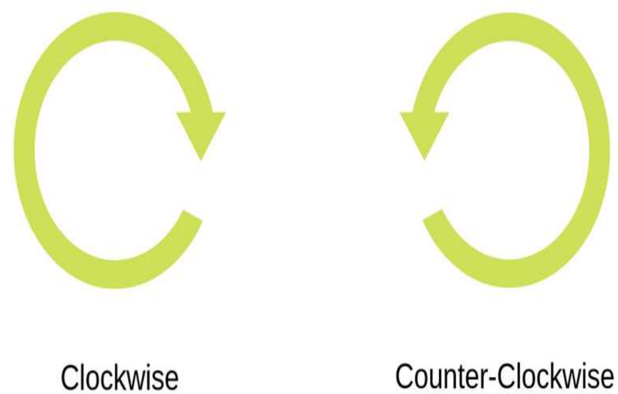
To describe a vector we need more information than to describe a scalar! Therefore vectors are more complex!

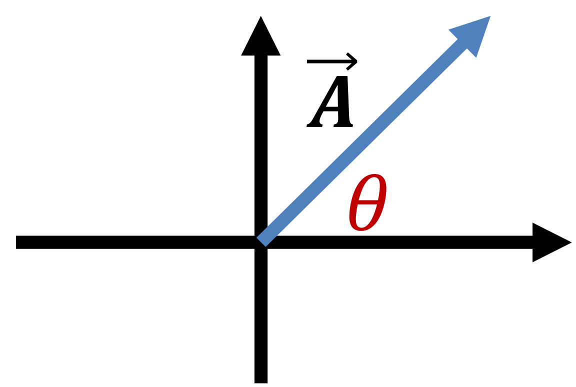
Important Notation

- To describe vectors we will use:
  - The bold font: Vector A is **A**
  - Or an arrow above the vector: $\vec{A}$
  - In the pictures, we will always show vectors as arrows
  - Arrows point the direction
  - To describe the magnitude of a vector we will use absolute value sign: $|\vec{A}|$ or just A,
  - The vector can be represented

    By its magnitude A, and its direction θ, (A, θ), where θ is the direction that the vector makes counterclockwise with the positive x- axis

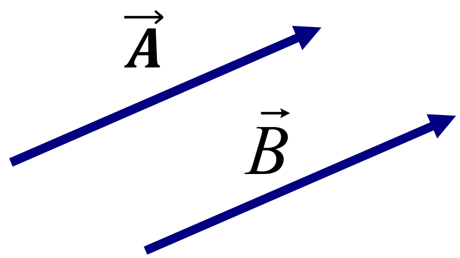


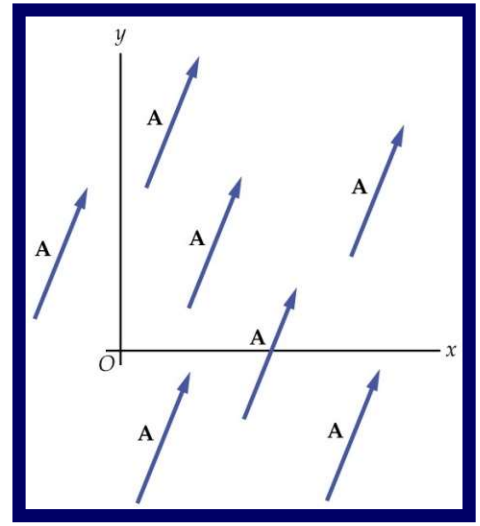




Properties of Vectors

- **Equality of Two Vectors**
  - **Two vectors are equal if they have the same magnitude and the same direction**
- **Movement of vectors in a diagram**
  - **Any vector can be moved parallel to itself without being affected**

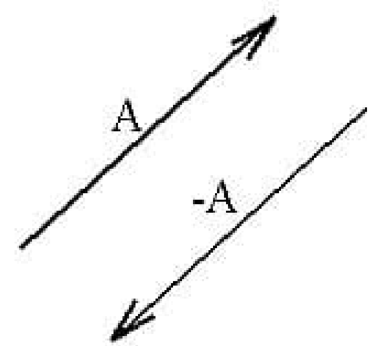




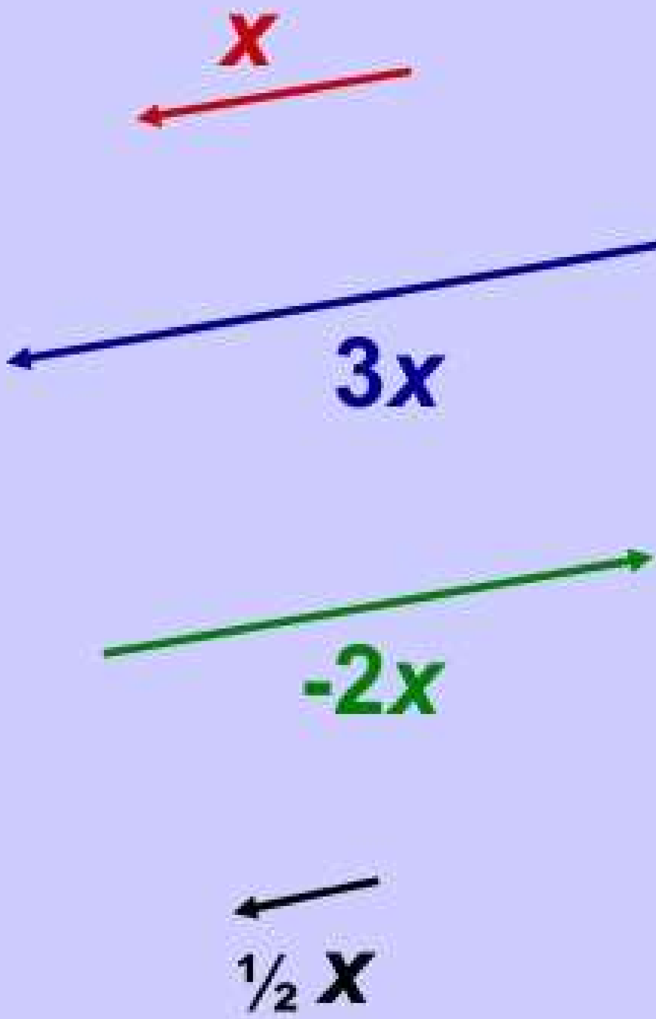
Negative Vectors

- Two vectors are **negative** if they have the same magnitude but are 180° apart (opposite directions)

$$\vec{\mathbf{A}} = -\vec{\mathbf{B}};\ \vec{\mathbf{A}} + \left(-\vec{\mathbf{A}}\right) = 0$$



Scalar Multiplication

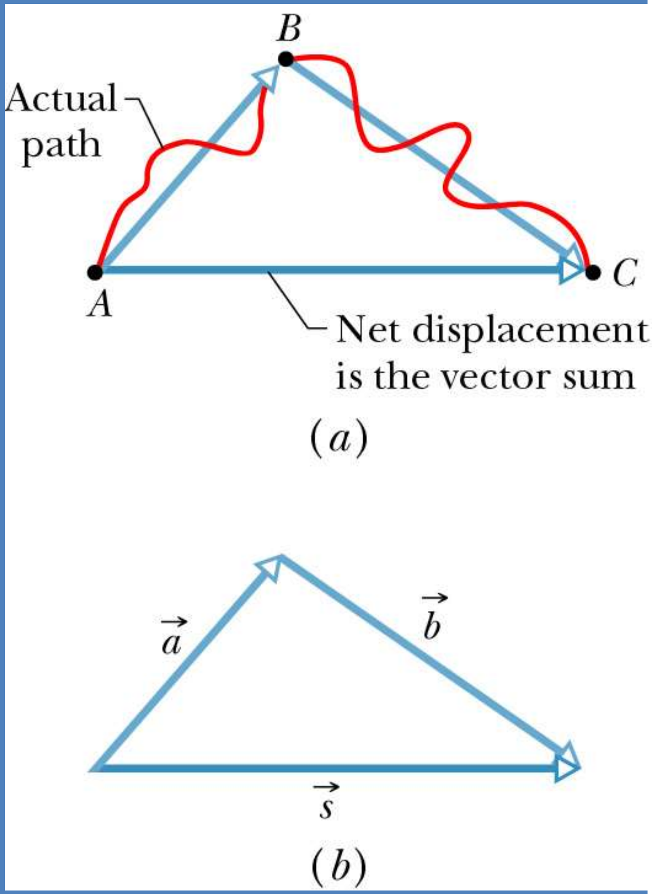


Scalar multiplication means multiplying a vector by a real number, such as 8.6. The result is a parallel vector of a different length. If the scalar is positive, the direction doesn't change. If it's negative, the direction is exactly opposite.

Blue is 3 times longer than red in the same direction. Black is half as long as red. Green is twice as long as red in the opposite direction.

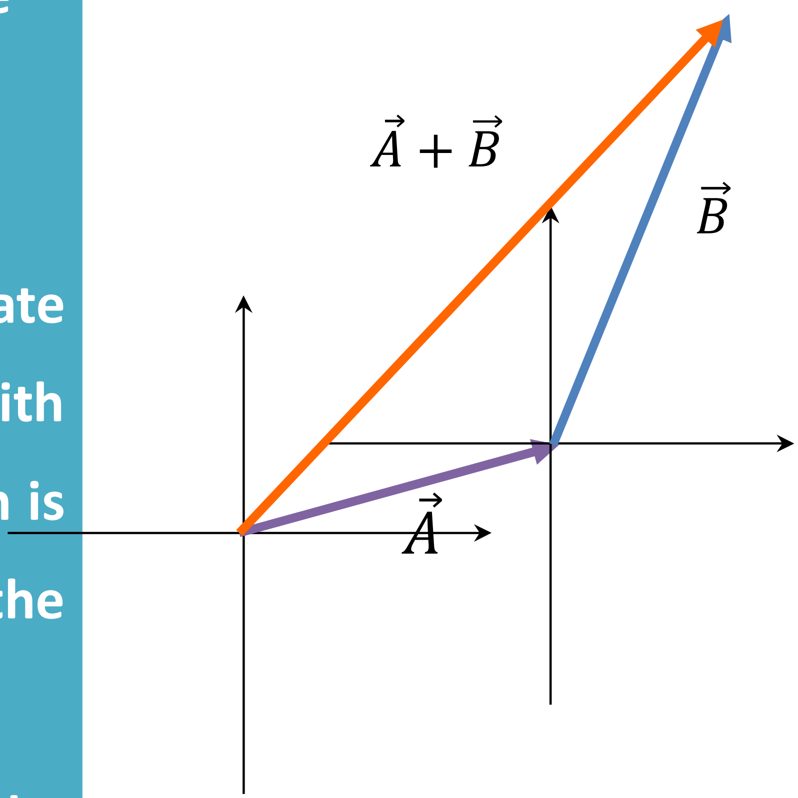
Adding Vectors (New Vector)

- When adding vectors, their directions must be taken into account
- Units must be the same
- Geometric Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient



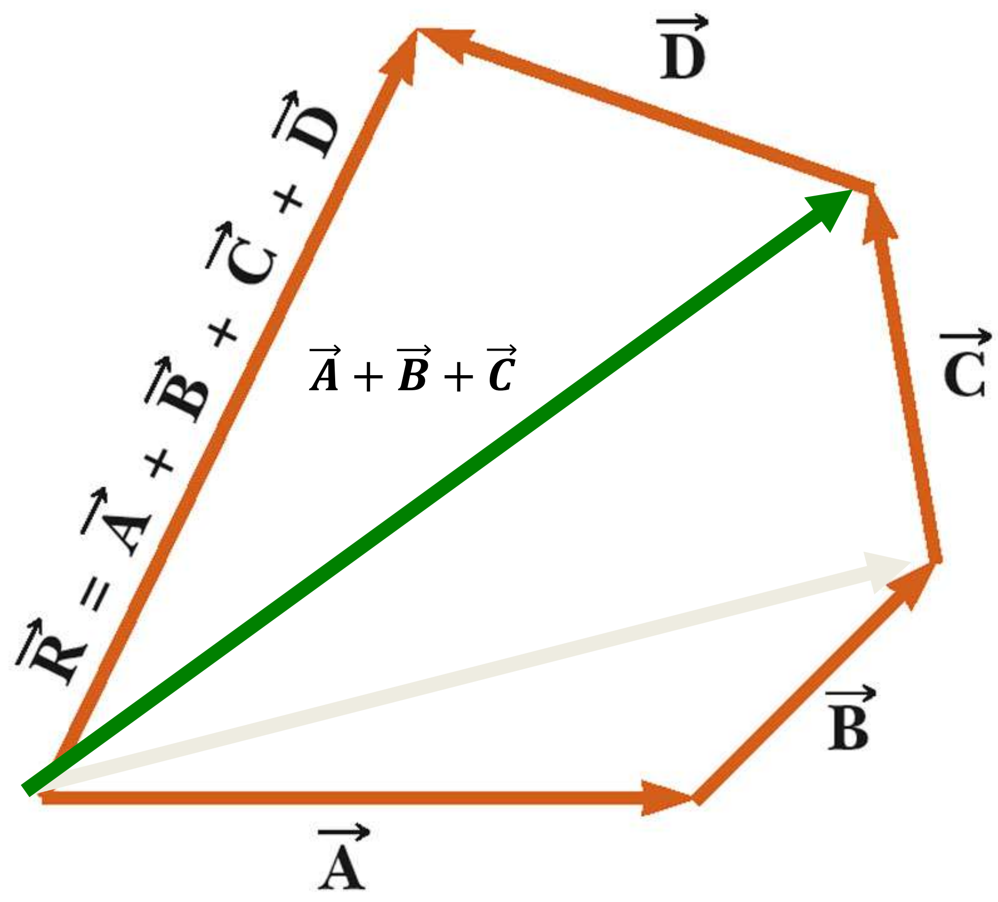
Adding Vectors Geometrically (Triangle Method)

- Draw the first vector $\vec{A}$ with the appropriate length and in the direction specified, with respect to a coordinate system
- Draw the next vector $\vec{B}$ with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector $\vec{A}$ and parallel to the coordinate system used for $\vec{A}$: “tip-to-tail”.
- The resultant is drawn from the origin of $\vec{A}$ to the end of the last vector $\vec{B}$



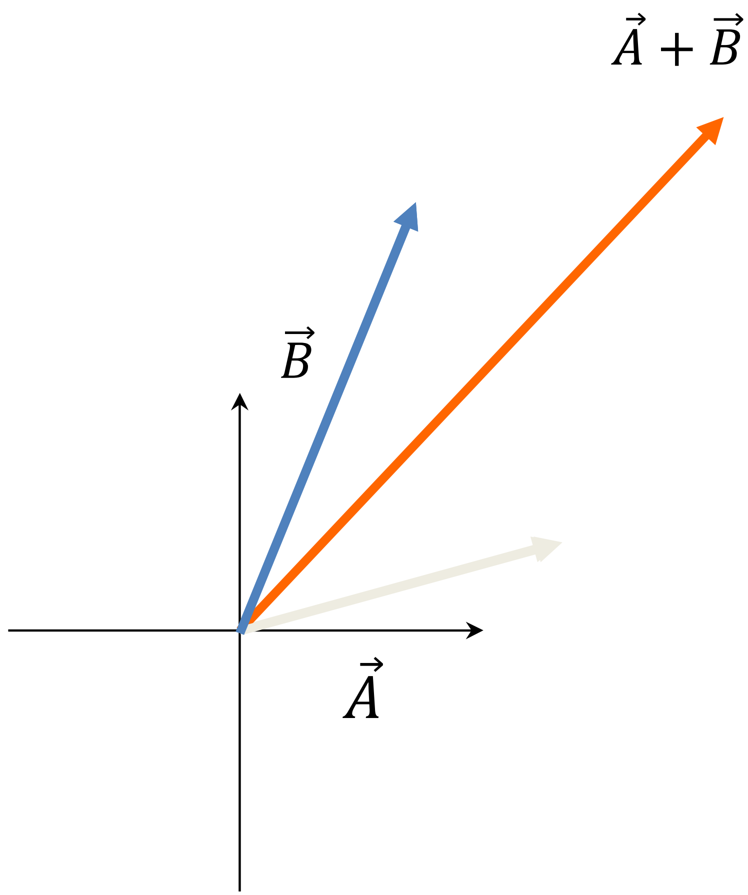
Adding Vectors Graphically

- **When you have many vectors, just keep repeating the process until all are included**
- **The resultant is still drawn from the origin of the first vector to the end of the last vector**



Adding vectors geometrically (polygon method)

- **Draw the first vector $\vec{A}$ with the appropriate length and in the direction specified, with respect to a coordinate system**
- **Draw the next vector $\vec{B}$ with the appropriate length and in the direction specified, with respect to the same coordinate system**
- **Draw a parallelogram**
- **The resultant is drawn as a diagonal from the origin**



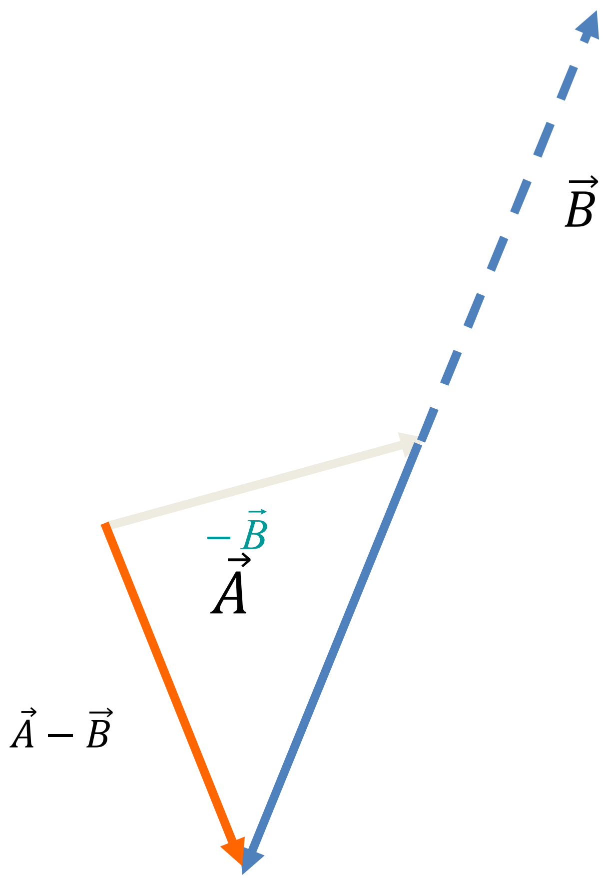
$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

Vector Subtraction

- Special case of vector addition
  - Add the negative of the subtracted vector

$$\vec{\mathbf{A}} - \vec{\mathbf{B}} = \vec{\mathbf{A}} + \left(-\vec{\mathbf{B}}\right)$$

- Continue with standard vector addition procedure

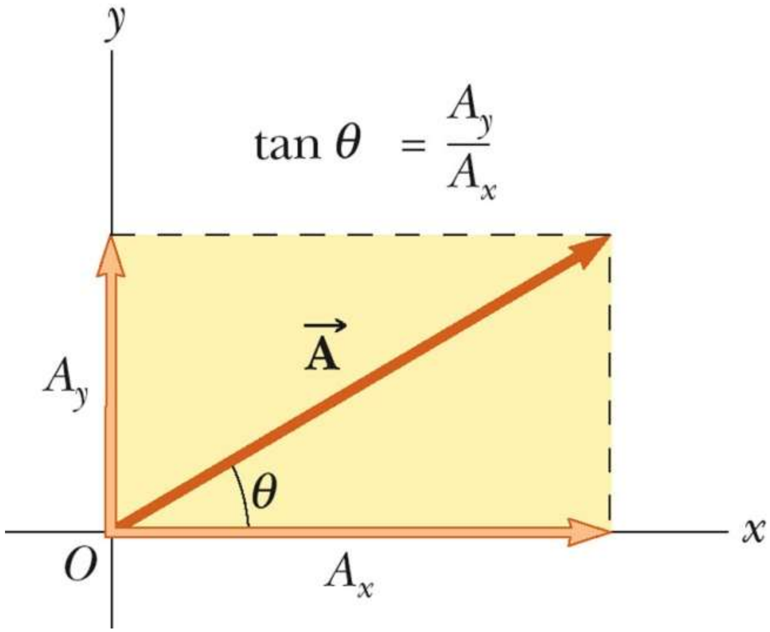


Vector Analysis
Components of a Vector

- A **component** is a part
- It is useful to use **rectangular components** These are the projections of the vector along the x- and y-axes

A_x is the x-component
A_y is the y-component

$$\cos\theta = \frac{A_x}{A} \qquad \sin\theta = \frac{A_y}{A}$$



$$\begin{cases} A_x = A\cos(\theta) \\ A_y = A\sin(\theta) \end{cases}$$

$$\begin{cases} \left|\vec{A}\right| = \sqrt{(A_x)^2 + (A_y)^2} \\ \tan(\theta) = \frac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \end{cases}$$

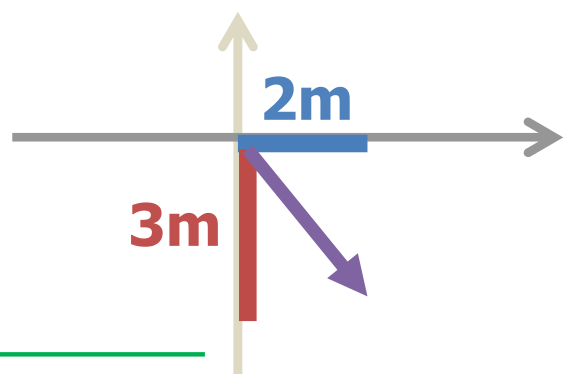
Components of a Vector

If we have a vector (4 m, 60°), *find the x and y component*

A_x = A cos θ = 4 cos 60 = 4 (0.5)= 2m

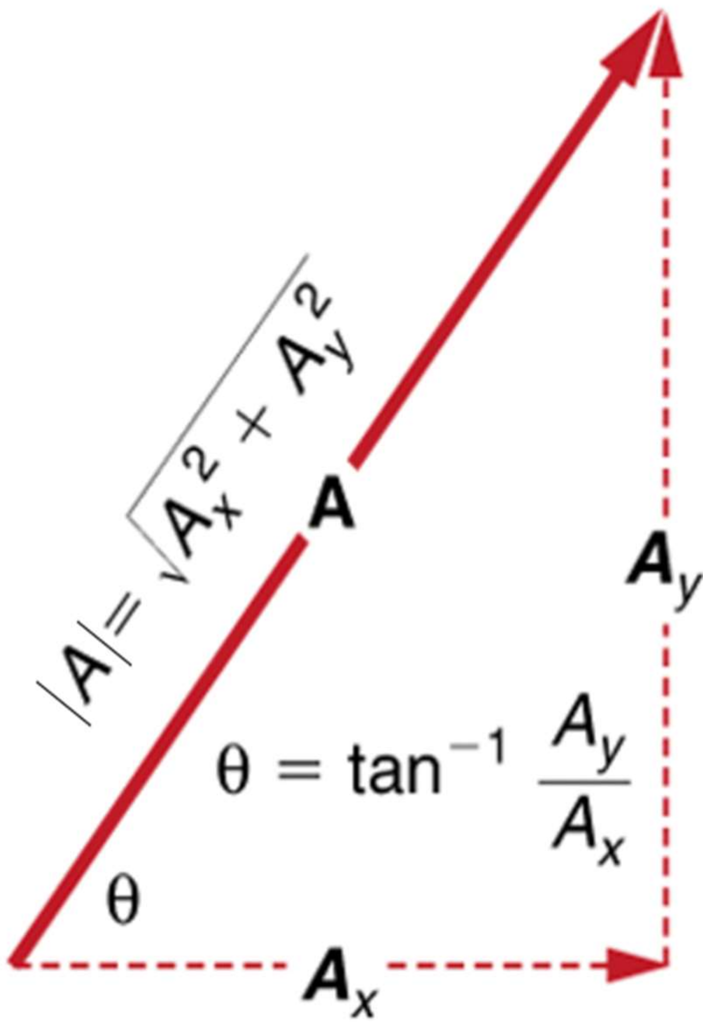
$A_y = A \sin 60 = 4(\frac{\sqrt{3}}{2}) = 2\sqrt{3}$ m

If we have a vector (2m, -3m), find its magnitude and direction



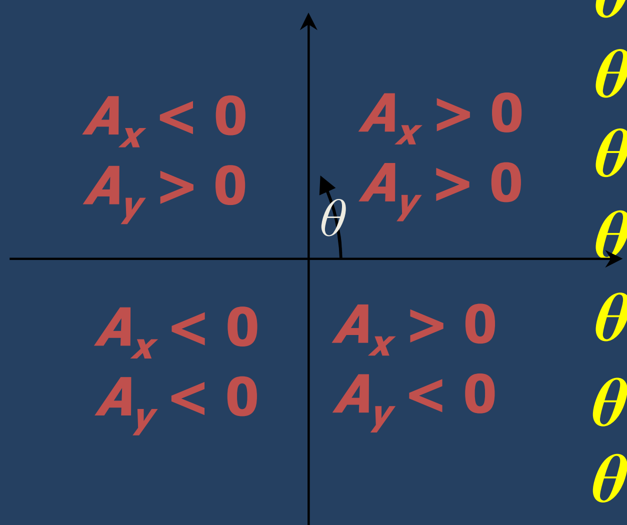
$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{4+9} = \sqrt{13}m$$

$$\theta = tan^{-1}(\frac{A_y}{A_x}) = tan^{-1}(\frac{-3}{2}) = -56.3^\circ = 303.7^\circ$$



Components of a Vector

- The previous equations are valid *only if ϑ is measured with respect to the x-axis*
- The components can be positive or negative and will have the same units as the original vector



$\theta=0,\ A_x=A>0,\ A_y=0$

$\theta=45^\circ,\ A_x=A\cos 45^\circ>0,\ A_y=A\sin 45^\circ>0$

$\theta=90^\circ,\ A_x=0,\ A_y=A>0$

$\theta=135^\circ,\ A_x=A\cos 135^\circ<0,\ A_y=A\sin 135^\circ>0$

$\theta=180^\circ,\ A_x=-A<0,\ A_y=0$

$\theta=225^\circ,\ A_x=A\cos 225^\circ<0,\ A_y=A\sin 225^\circ<0$

$\theta=270^\circ,\ A_x=0,\ A_y=-A<0$

$\theta=315^\circ,\ A_x=A\cos 315^\circ<0,\ A_y=A\sin 315^\circ<0$

Addition and subtraction of Vectors

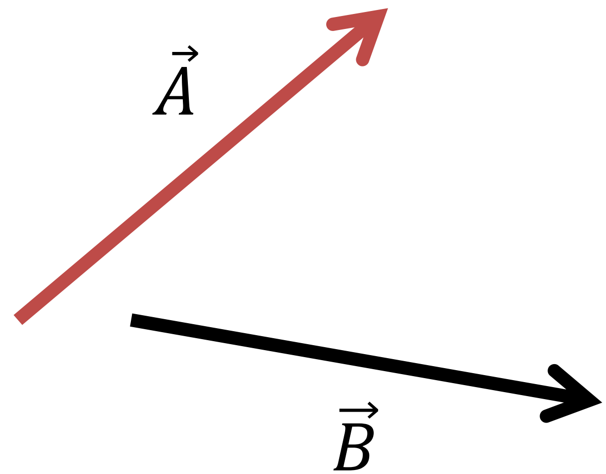
$\vec{A}$: (A_x, A_y) $\vec{B}$: (B_x, B_y)

$\vec{A} \mp \vec{B} = \vec{R}$, where $\vec{R}$: (R_x, R_y)

$R_x = A_x \mp B_x$,

$R_y = A_y \mp B_y$

$$\boldsymbol{R = \sqrt{R_x^2 + R_y^2}}$$



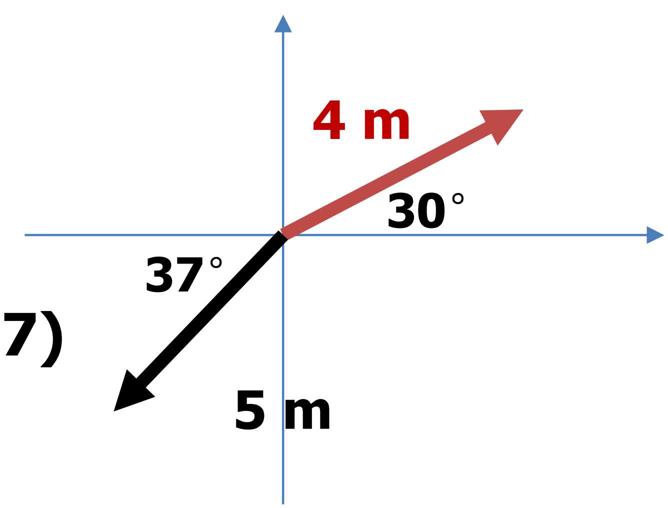
Two vectors $\vec{A}$: (4 m, 30°) and $\vec{B}$: (5 m, 217°), find $\vec{A} - \vec{B}$

$\vec{A}$: (4 m, 30°)= (4cos30, 4sin30)

$\vec{B}$: (5 m, 217°) =(- 5cos37, -5sin37)

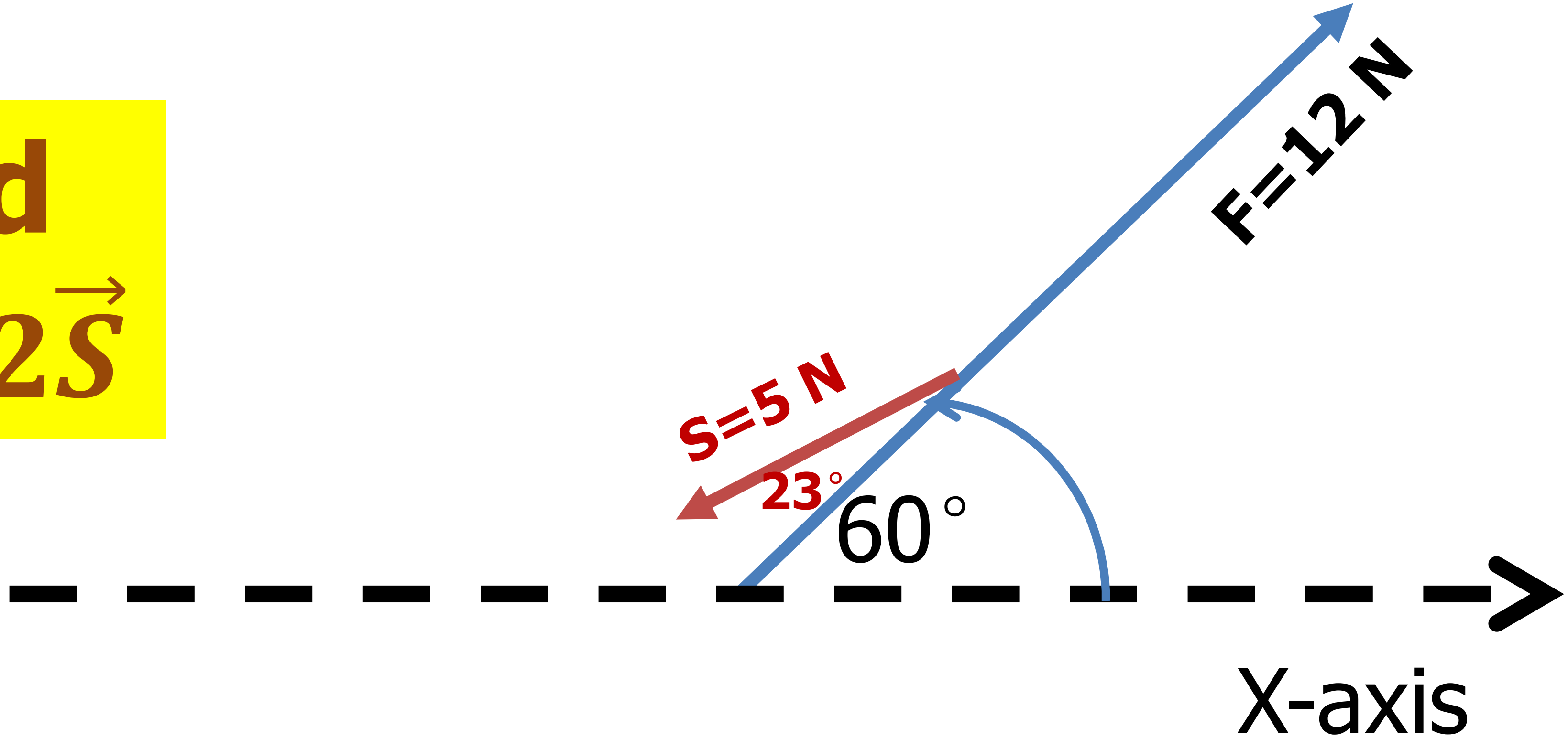
$\vec{A} - \vec{B} = \vec{R}$: ($R_x$, R_y)

= (4cos30+5cos37, 4sin30+ 5sin37)



Find

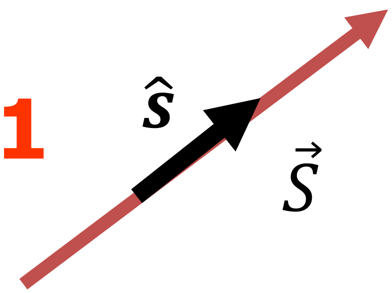
$$\vec{F} + 2\vec{S}$$



Unit Vector

A **unit vector** is a vector with a magnitude of one unit. Any vector has an associated unit vector in the same direction but having a magnitude of one. Any vector can be expressed as a [scalar multiple] of its unit vector.

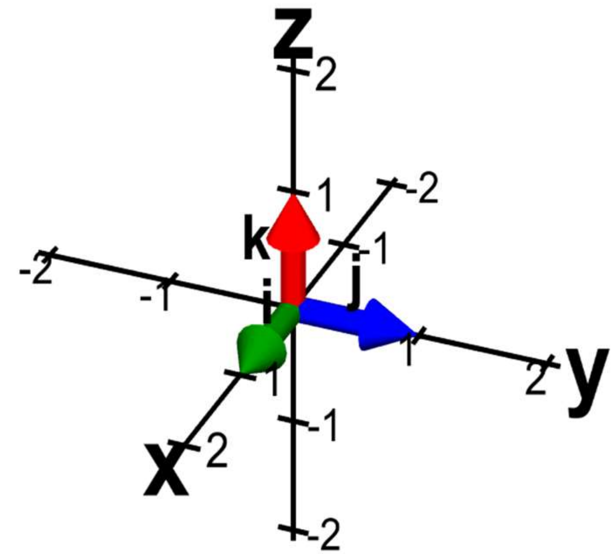
$\hat{s} = \frac{\vec{s}}{s}$, and $|\hat{s}| = \frac{|\vec{s}|}{s} = 1$



- Unit vectors *i*-hat, *j*-hat, *k*-hat

  $\hat{i} \rightarrow x \quad \hat{j} \rightarrow y \quad \hat{k} \rightarrow z$
- Unit vectors used to specify direction
- Unit vectors have a magnitude of 1
- Then $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



Example : Operations with Vectors

- Vector **A** is described algebraically as **(-3, 5),** while vector **B** is **(4, -2).** Find the value of **magnitude** and **direction** of the sum (**C**) of the vectors **A** and **B**.

$$\vec{A} = -3\hat{\imath} + 5\hat{\jmath} \qquad \vec{B} = 4\hat{\imath} - 2\hat{\jmath}$$

$$\vec{C} = \vec{A} + \vec{B} = (-3+4)\hat{\imath} + (5-2)\hat{\jmath} = 1\hat{\imath} + 3\hat{\jmath}$$

$$C_x = 1 \qquad C_y = 3$$

$$C = (C_x{}^2 + C_y{}^2)^{1/2} = (1^2 + 3^2)^{1/2} = 3.16$$

$$\theta = \tan^{-1}\frac{C_y}{C_x} = \tan^{-1} 3 = 71.56^\circ$$

Example

If we have two vectors $\vec{C} = 2i - 3j + 5k$ *and* $\vec{D} = -i + 5j - k.$ *find*
a) $\vec{C} + 2\vec{D}$**,** **b) the unit vector in the direction of** $\vec{D} + 2\vec{C}$

a)

$$\vec{C} + 2\vec{D} = \vec{M}$$

where $\vec{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$

$$M_x = C_x + 2D_x = 2 + (-2) = 0$$

$$M_y = C_y + 2D_y = -3 + (10) = 7$$

$$M_z = C_z + 2D_z = 5 + (-2) = 3$$

$$\therefore \vec{C} + 2\vec{D} = \boxed{\vec{M} = 7\hat{j} + 3\hat{k}}$$

b)

$D + 2\vec{C} = \vec{F}$, $\vec{F} = (-\hat{i} + 4\hat{i}) + (5\hat{j} - 6\hat{j}) + (-\hat{k} + 10\hat{k})$

$\vec{F} = -5\hat{i} + \hat{j} + 9\hat{k}$, $F = \sqrt{25 + 1 + 81} = \sqrt{107}$

unit vector $\hat{f} = \dfrac{\vec{F}}{F} = \dfrac{1}{\sqrt{107}}(-5\hat{i} - \hat{j} + 9\hat{k})$

Adding Vectors Algebraically

- Consider two vectors

$$\vec{\boldsymbol{A}} = \boldsymbol{A_x}\hat{\boldsymbol{\imath}} + \boldsymbol{A_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{A_z}\widehat{\boldsymbol{k}}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + \boldsymbol{B_z}\widehat{\boldsymbol{k}}$$

- Then

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \left(\boldsymbol{A_x}\hat{\boldsymbol{\imath}} + \boldsymbol{A_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{A_z}\widehat{\boldsymbol{k}}\right) + \left(\boldsymbol{B_x}\hat{\boldsymbol{\imath}} + \boldsymbol{B_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{B_z}\widehat{\boldsymbol{k}}\right)$$
$$= (\boldsymbol{A_x} + \boldsymbol{B_x})\hat{\boldsymbol{\imath}} + (\boldsymbol{A_y} + \boldsymbol{B_y})\hat{\boldsymbol{\jmath}} + (\boldsymbol{A_z} + \boldsymbol{B_z})\widehat{\boldsymbol{k}}$$

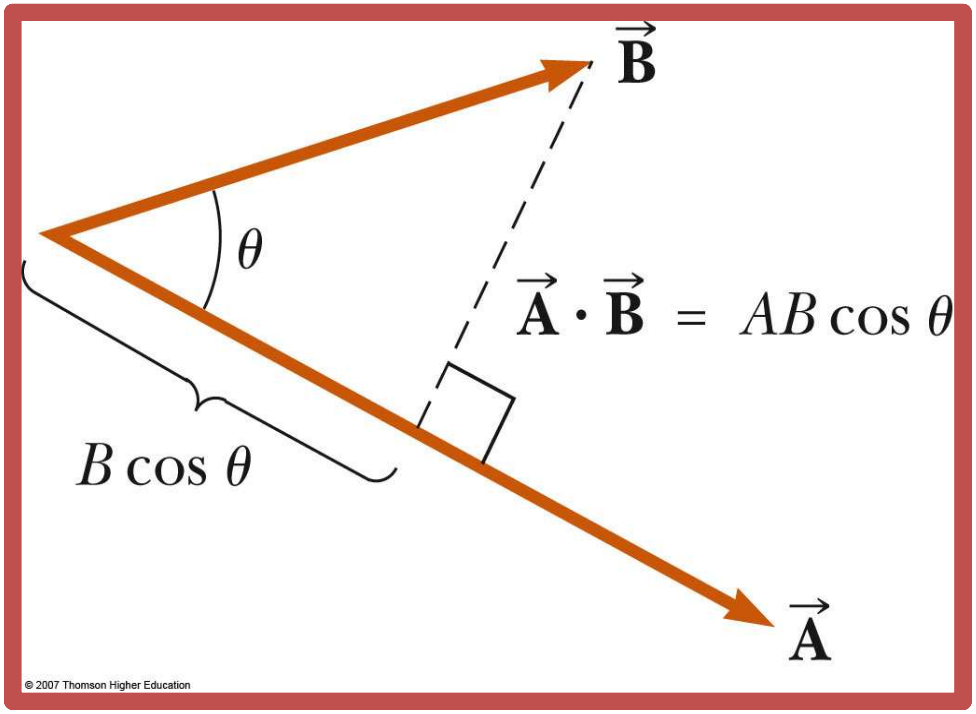
- If $\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (\boldsymbol{A_z} + \boldsymbol{B_z})\widehat{\boldsymbol{k}}$
- so $C_x = A_x + B_x \qquad C_y = A_y + B_y$

$$C_z = (\boldsymbol{A_z} + \boldsymbol{B_z})$$

Scalar Product of Two Vectors

The scalar product, or dot product, is a mathematical operation used to determine the component of a given force in the direction of the displacement

- **The scalar product of two vectors is written as** $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$
  - **It is also called the dot product**
- $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv A\, B\, cos\, \theta$
  - θ **is the angle *between* *A* and *B***
- **Applied to work, this means**



$$W = F\Delta r \cos\theta = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}}$$

Properties of Scalar Products

If	Then
$\vec{A}$ and $\vec{B}$ are $\perp$	$\vec{A}\cdot\vec{B}=0$
$\vec{A}$ and $\vec{B}$ are parallel	$\vec{A}\cdot\vec{B}=AB$
$\vec{A}\cdot\vec{B}=0$	Either $\vec{A}=0$ or $\vec{B}=0$ or $\vec{A}\perp\vec{B}$

Furthermore,

$\vec{A}\cdot\vec{A}=A^2$ Because $\vec{A}$ is parallel to itself

$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$ Commutative rule of multiplication

$(\vec{A}+\vec{B})\cdot\vec{C}=\vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}$ Distributive rule of multiplication

For unit vectors

$\hat{\imath}.\hat{\imath}=\hat{\jmath}.\hat{\jmath}=\hat{k}.\hat{k}=1$, **each unit vector parallel with itself**

$\hat{\imath}.\hat{\jmath}=\hat{\jmath}.\hat{k}=\hat{\imath}.\hat{k}=0$, **as they are perpendicular to each other**

Dot Product

- The dot product says something about how parallel two vectors are.
- The dot product (scalar product) of two vectors can be thought of as the projection of one onto the direction of the other.

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$
$$\vec{A} \cdot \hat{\imath} = A\cos\theta = A_x$$

- Components

$$\boldsymbol{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

Derivation

- How do we show that $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$
- Start with $$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$ $$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then $$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\end{aligned}$$
- But $$\hat{i}\cdot\hat{j} = 0;\ \hat{i}\cdot\hat{k} = 0;\ \hat{j}\cdot\hat{k} = 0$$ $$\hat{i}\cdot\hat{i} = 1;\ \hat{j}\cdot\hat{j} = 1;\ \hat{k}\cdot\hat{k} = 1$$
- So $$\begin{aligned}\vec{A}\cdot\vec{B} &= A_x\hat{i}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_y\hat{j} + A_z\hat{k}\cdot B_z\hat{k} \\ &= A_x B_x + A_y B_y + A_z B_z\end{aligned}$$

Example 1.17

Consider the two vectors **a** and **b**. Suppose **a** has modulus 4 units, **b** has modulus 5 units, and the angle between them is 60° . Find the scalar product of the two vectors.

Example 1.19

Suppose we wish to test whether or not the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, where

$\mathbf{a} = 3i + 2j - k$ and $\mathbf{b} = i - 2j - k$

Given the vectors: **A** = 3**i** + 2**j** − **k** and **B** = 5**i** + 5**j**, find:

a. The dot product **A**·**B**.

b. The projection of **A** onto **B**.

c. The angle between **A** and **B**.

d. A vector of magnitude 2 in the XY plane perpendicular to **B**.

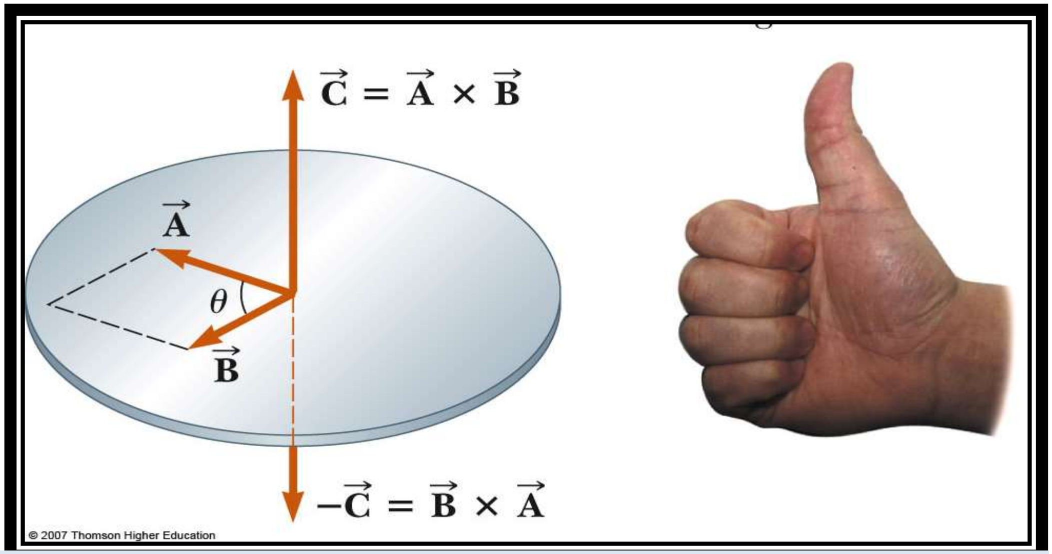
e. Find the angles that the vector A+B makes with the X,Y and Z axises

Cross Product

The vector product or cross product of two vectors is defined as another vector having a magnitude equal to the product of the magnitudes of two vectors and the sine of the angle between them. The direction of the product vector is perpendicular to the plane containing the two vectors, in accordance with the right hand screw rule or right hand thumb rule

$$\vec{C} = \vec{A} \times \vec{B}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = AB \sin\theta$$



- **Direction: C perpendicular to both A and B First practice**

$$\vec{A} \times \vec{B} = \vec{B} \times \vec{A}\ ?$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

The cross product of two vectors results in a new vector.

- θ is smaller angle between the vectors
- Cross product of any parallel vectors = zero
- Cross product is maximum for perpendicular vectors

Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

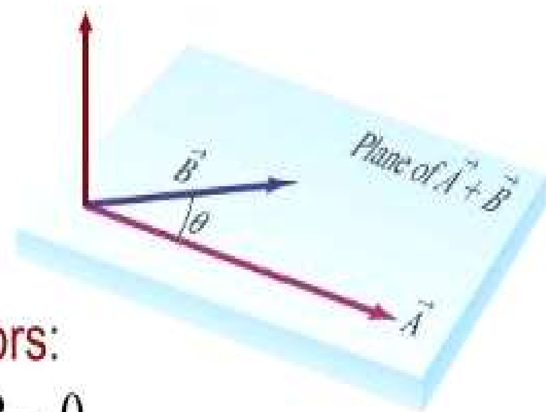
- Cross product:

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} \Rightarrow C = AB\sin\theta; \quad \mathbf{C} \perp \mathbf{A}; \quad \mathbf{C} \perp \mathbf{B}$$

- Properties:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$



- Parallel and orthogonal vectors:

$$\mathbf{A} \parallel \mathbf{B} \Rightarrow \sin\theta = 0 \Rightarrow \mathbf{A} \times \mathbf{B} = 0$$

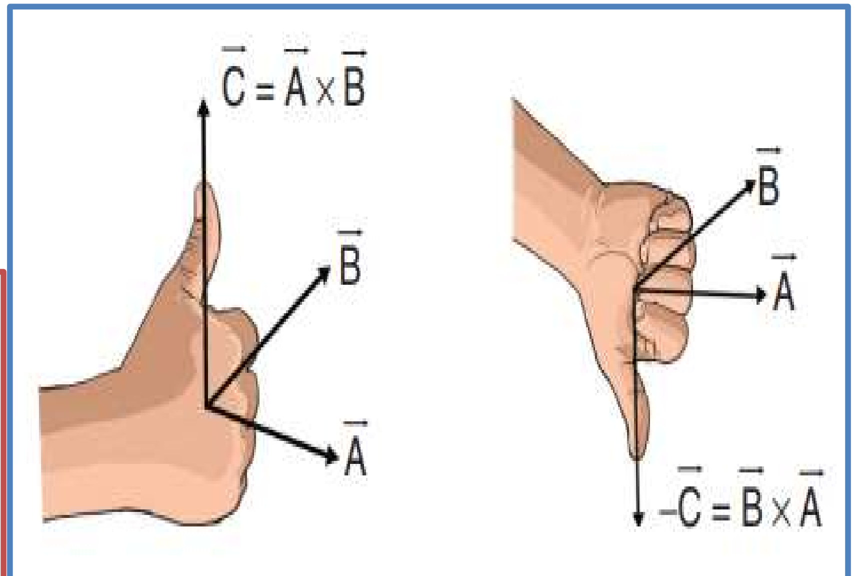
$$\mathbf{A} \perp \mathbf{B} \Rightarrow \sin\theta = 1 \Rightarrow |\mathbf{A} \times \mathbf{B}| = AB$$

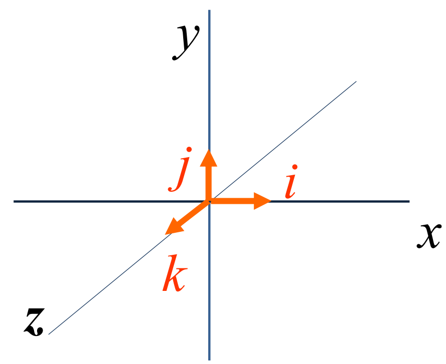
- Cross product of a vector with itself:

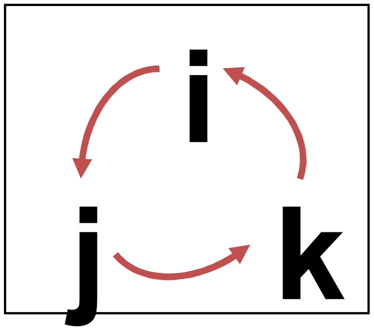
$$\mathbf{A} \times \mathbf{A} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}; \quad \hat{i} \times \hat{k} = -\hat{j}; \quad \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{i} \times \hat{i} = 0; \quad \hat{j} \times \hat{j} = 0; \quad \hat{k} \times \hat{k} = 0$$







More about Cross Product

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$

- How do we show that $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$
- Start with

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

- Then

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_x\hat{\imath} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_y\hat{\jmath} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_z\hat{k} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})\end{aligned}$$

- But

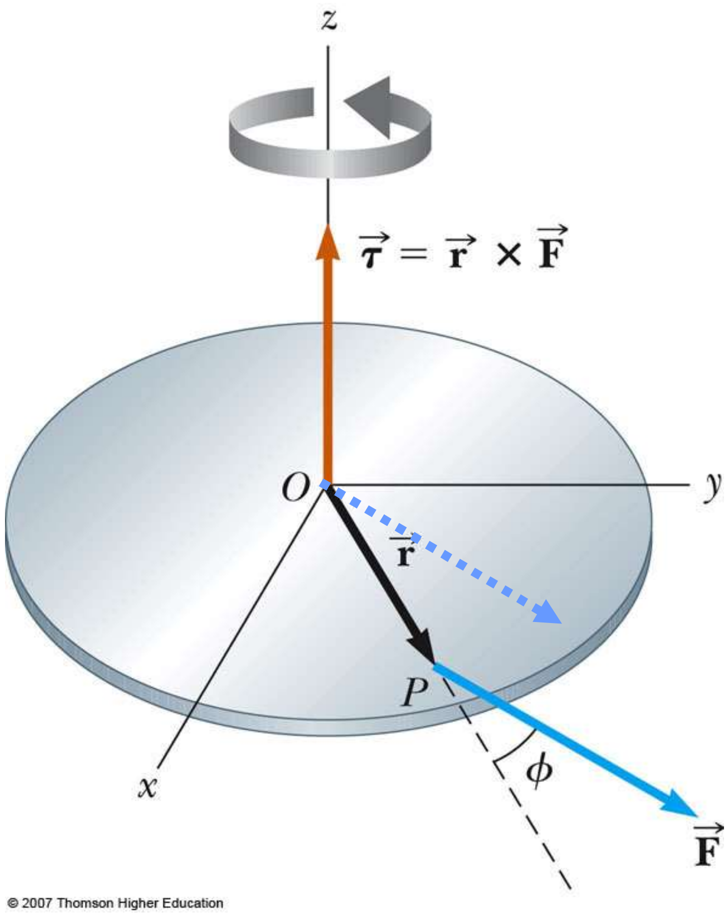
$$\hat{\imath} \times \hat{\jmath} = \hat{k};\ \ \hat{\imath} \times \hat{k} = -\hat{\jmath};\ \ \hat{\jmath} \times \hat{k} = \hat{\imath}$$
$$\hat{\imath} \times \hat{\imath} = 0;\ \ \hat{\jmath} \times \hat{\jmath} = 0;\ \ \hat{k} \times \hat{k} = 0$$

- So

$$\vec{A} \times \vec{B} = A_x\hat{\imath} \times B_y\hat{\jmath} + A_x\hat{\imath} \times B_z\hat{k} + A_y\hat{\jmath} \cdot B_x\hat{\imath} + A_y\hat{\jmath} \cdot B_z\hat{k}$$
$$+A_z\hat{k} \cdot B_x\hat{\imath} + A_z\hat{k} \cdot B_y\hat{\jmath}$$

Torque as a Cross Product

$$\vec{\tau} = \vec{r} \times \vec{F}$$



- The torque is the cross product of a force vector with the position vector to its point of application

$$|\tau| = rF\sin\theta = r_{\perp}F = r\,F_{\perp}$$

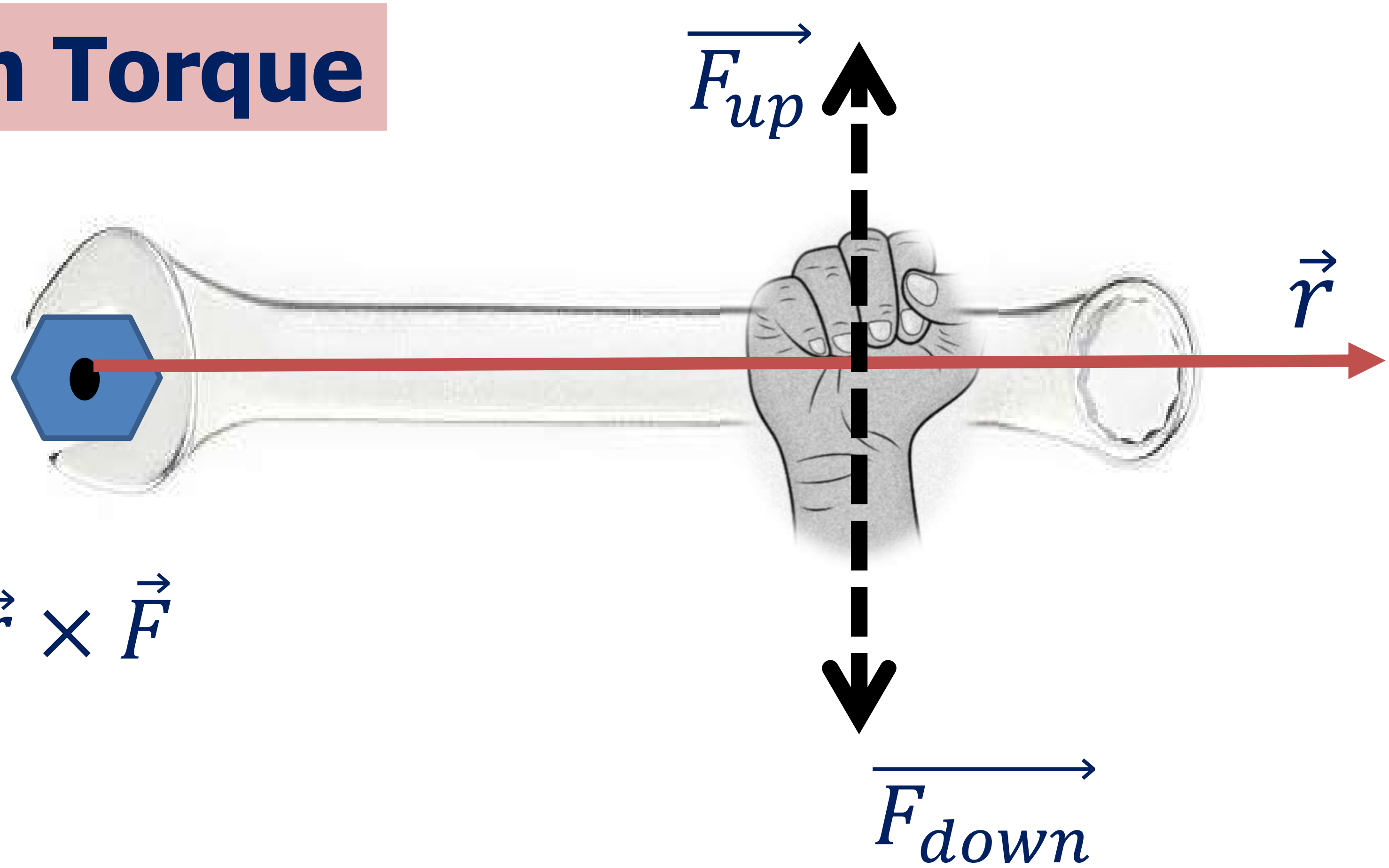
- The torque vector is perpendicular to the plane formed by the position vector and the force vector (e.g., imagine drawing them tail-to-tail)
- Right Hand Rule: curl fingers from r to F, thumb points along torque.

$$\vec{\tau}_{net} = \sum_{all\ i} \vec{\tau}_i = \sum_{all\ i} \vec{r}_i \times \vec{F}_i \ \ (\text{vector sum})$$

Superposition:

- **Can have multiple forces applied at multiple points.**
- **Direction of τ_{net} is angular acceleration axis**

Wrench Torque



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\overrightarrow{F_{up}}$ cause a counterclockwise rotation, then the torque is positive

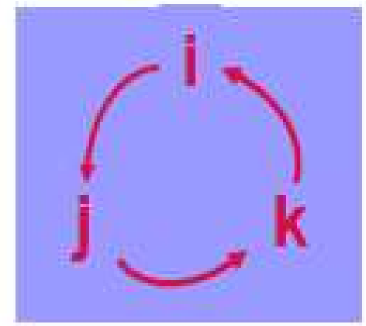
$\overrightarrow{F_{down}}$ cause a clockwise rotation, then the torque is negative

Calculating Cross Products

Find: $\vec{A}\times\vec{B}$ **Where:** $\vec{A}=2\hat{i}+3\hat{j}$ $\vec{B}=-\hat{i}+2\hat{j}$

Solution:

$$\vec{A}\times\vec{B}=(2\hat{i}+3\hat{j})\times(-\hat{i}+2\hat{j})$$
$$=2\hat{i}\times(-\hat{i})+2\hat{i}\times2\hat{j}+3\hat{j}\times(-\hat{i})+3\hat{j}\times2\hat{j}$$
$$=0+4\hat{i}\times\hat{j}-3\hat{j}\times\hat{i}+0=4\hat{k}+3\hat{k}=7\hat{k}$$



Calculate torque given a force and its location

$$\vec{F}=(2\hat{i}+3\hat{j})N \qquad \vec{r}=(4\hat{i}+5\hat{j})m$$

Solution:

$$\vec{\tau}=\vec{r}\times\vec{F}=(4\hat{i}+5\hat{j})\times(2\hat{i}+3\hat{j})$$
$$=4\hat{i}\times2\hat{i}+4\hat{i}\times3\hat{j}+5\hat{j}\times2\hat{i}+5\hat{j}\times3\hat{j}$$
$$=0+4\hat{i}\times3\hat{j}+5\hat{j}\times2\hat{i}+0=12\hat{k}-10\hat{k}=2\hat{k}\ (\mathrm{Nm})$$

Determinent Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{\imath} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + (-\hat{\jmath}) \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\begin{aligned} \vec{A} \times \vec{B} = \; & (A_y B_z - A_z B_y)\,\hat{\imath} \quad + \\ & (A_z B_x - A_x B_z)\,\hat{\jmath} \quad + \\ & (A_x B_y - A_y B_x)\,\hat{k} \end{aligned}$$

Two vectors u= i+2j+k, and v= -j+2k, find uxv

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} - \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= i\,(2x2 - 1x(-1)) - j\,(1x2 - 1x0) + k(1x(-1) - 2x0)$$

$$= 5i - 2j - 1k = (5, -2, -1)^T$$

Given the vectors $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ find $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$.

Answer 1:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \mathbf{i}(4 - 4) + \mathbf{j}(12 + 2) + \mathbf{k}(1 + 6)$$

$$= 14\mathbf{j} + 7\mathbf{k}$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{196 + 49} = \sqrt{245} = 15.7$$

24. For the following three vectors $\mathbf{A} = -i + 2j$, $\mathbf{B} = i - 2k$ *,and* $\mathbf{C} = -2j + 2k$ find,

1. The angle between the vectors $\mathbf{D} = \mathbf{A} + \mathbf{B}$, *and* $\mathbf{T} = \mathbf{A} - \mathbf{C}$

2. find the vectors $\mathbf{E} = \mathbf{A} \times \mathbf{C}$, *and* $\mathbf{F} = \mathbf{B} \times \mathbf{C}$, and the unit vectors $\hat{E}$

25. When two vectors A and B are drawn from a common point, the angle between them is θ . Show that the magnitude of their vector sum is given by $\sqrt{A^2 + B^2 + 2AB\cos\theta}$. If **A** and **B** have the same magnitude, under what condition will their vector sum have the same have the magnitude of the vector **A** or **B**?

Two vectors $\vec{A}$ and $\vec{B}$ have the magnitudes of 2 and 5 respectively, and the magnitude of their vector product is 6. Find the magnitude of their scalar product.

A) A sailboat sails 2 km east, then 4 km southeast, and then an additional unknown displacement. Its final position is 6 km directly east of the starting point. Find the third **unknown displacement** of the journey.

B) Two vectors $\vec{A} = \hat{i} - 3\hat{k}$, $\vec{B} = 3\hat{j} - \hat{k}$, find the **unit vector** in the direction of $\vec{B} \times \vec{A}$

If $\vec{A} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{B} = \hat{i} + \hat{j}$, find

a) a third vector $\vec{C}$ has the magnitude of $\vec{A}$ and make an angle 90° to the vector $\vec{B}$.

b) The angle between $(\vec{A} \times \vec{B})$ *and* $(\vec{A} - \vec{C})$

Three vectors A= 2m, B= 4m and C= 3m. Vector A lies in the x-z plane make an angle 30° with the z-axis, Vector B lies in the y-z plane make an angle 60° with the y-axis and Vector C lies parallel to the y-axis. Find

a) $\vec{B}\cdot(\vec{A}\times\vec{C})$

b) The angle that the vector $\vec{A}-\vec{C}$ make with the z-axis

Vector $\vec{A}=2\hat{\imath}+3\hat{\jmath}-4\hat{k}$ and vector $\vec{B}=-2\hat{\imath}+3\hat{\jmath}-7\hat{k}$, Find a third vector in the 3D-space has double of magnitude and perpendicular to the vector $(\vec{A}-\vec{B})$ and make an angle 60° with the Y- axis.

Two vectors $\vec{A}$ and $\vec{B}$ have the same length and located in the xz plane are drawn from a common point. The Vector $\vec{B}$ is 60° above vector $\vec{A}$. If their vector sum equal 4 m, what is the magnitude of each of them. Find the unit vector in the direction perpendicular to $\vec{A}$ and $\vec{A}+\vec{B}$.